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A user's guide to the local arithmetic of hyperelliptic curves

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Abstract

A new approach has been recently developed to study the arithmetic of hyperelliptic curves $y^2 = f(x)$ over local fields of odd residue characteristic via combinatorial data associated to the roots of f. Since its introduction, numerous papers have used this machinery of 'cluster pictures' to compute a plethora of arithmetic invariants associated to these curves. The purpose of this user's guide is to summarise and centralise all of these results in a self-contained fashion, complemented by an abundance of examples.

MSC (2020)
11G20 (primary), 11G10, 14D10, 14G20, 14H45, 14Q05 (secondary)

Contents

1 | INTRODUCTION

In this paper, we provide a summary of a recently developed approach to understanding the local arithmetic of hyperelliptic curves. This approach revolves around the theory of 'clusters', and enables one to read off many local arithmetic invariants of hyperelliptic curves from explicit equations $y^2 = f(x)$. The paper is meant to serve as a *user's* guide: our aim has been to make it accessible to mathematicians interested in applications outside of local arithmetic geometry, or who may wish to compute local invariants without having to decipher the theoretical background.

Throughout this article, K will be a local field of odd residue characteristic p and C/K a hyperelliptic curve given by

$$y^2 = f(x) = c\prod_{r\in\mathcal{R}}(x-r),$$

where $f \in K[x]$ is separable, $\deg(f) = 2g+1$ or $2g+2$ and $g \geqslant 2$.

1.1 | How to use this guide

The article is structured as follows. We begin in Section 2 by declaring some general notation which will be used throughout, and proceed to give some background theory on cluster pictures and BY trees in Sections 3 and 4, respectively. Cluster pictures will be critical background for all sections of the article; BY trees will be used in Sections 10, 15, 17 and the Appendix.

From there on, each section will be self-contained and independent of the other sections. This will allow a reader who is concerned with just one topic (Galois representations, say) to be able to learn everything they need by reading just the background theory in Sections 3 and 4 and the relevant section (in our example, Section 11).

From Section 5 onwards, each section will consist of two parts: the first stating the relevant theorems, and the second providing examples illustrating the theorems. None of the theorems are original (apart from Theorem A.6, whose proof is given in the Appendix) and we give no proofs; each section has references at the end where the interested reader can find proofs and more general statements of the theorems.

1.2 | Related work

The key references for the present work are [3, 4, 9, 10, 14, 15, 20]. We have made a blanket assumption that K is a local field; this is often unnecessarily restrictive, and many results hold for complete discretely valued fields. The reference [9] also discusses a number of topics that we have omitted, in particular how to use clusters to check whether a curve is deficient, how one may perturb $f(x)$ without changing the standard local invariants, and how to classify semistable hyperelliptic curves in a given genus.

As many of our examples will illustrate, the method of cluster pictures is very convenient for computations. However, it can also be used for more theoretical purposes: for instance, one can work explicitly with families of hyperelliptic curves for which the genus becomes arbitrarily large (see, for example, [1, 7]), or prove general results for curves of a given genus by a complete case-by-case analysis of cluster pictures (see, for example, [13]).

We would like to mention some alternative techniques that have been recently developed for investigating similar topics. In [6, 16, 21–23], the authors determine different kinds of models, the conductor exponent, the local L-factor, compare the Artin conductor to the minimal discriminant and compute a basis of the integral differentials. In arbitrary residue characteristic (including 2), but under some technical assumptions, [8, 12, 20] determine the minimal regular model with normal crossings, a basis of integral differentials, reduction types, conductor and action of the inertia group on the ℓ-adic representation.

1.3 | Implementation

We have implemented many of the methods described in this guide as a package using the SageMath computer algebra system [24]. The package is available online at [2]. This package includes implementations of cluster pictures and BY trees as abstract objects, which it can also plot. Given a hyperelliptic curve, the implementation determines its associated cluster picture and BY tree. It also determines the Tamagawa number, root number, reduction type, minimal discriminant and dual graph of the minimal regular model, as described in this article.

We have also computed cluster pictures for all elliptic curves over $\mathbb{Q}$ and number fields, and all genus 2 curves present in the L-Functions and Modular Forms Database [19]. The latter is incorporated in the LMFDB homepages of curves.

2 | NOTATION

Here we set out the notation that will be used throughout the paper.

Formally by a hyperelliptic curve C we mean the smooth projective curve associated to $y^2 = f(x)$, equivalently the gluing of the pair of affine patches

$$y^2 = f(x) \quad \text{and} \quad v^2 = t^{2g+2} f\left(\frac{1}{t}\right)$$

along the maps $x = \frac{1}{t}$ and $y = \frac{v}{t^{g+1}}$, where $f \in K[x]$ is separable, and $\deg(f) \geqslant 5$. We will not consider double covers of general conics.

We fix the following notation associated to fields and hyperelliptic curves.

K	local field of odd residue characteristic p
$\mathcal{O}_K$	ring of integers of K
k	residue field of K
π	uniformiser of K
v	normalised valuation with respect to K so that $v(\pi) = 1$
$\bar{K}$	algebraic closure of K
K^{sep}	separable closure of K inside $\bar{K}$
K^{nr}	maximal unramified extension of K inside K^{sep}
$\bar{k}$	algebraic closure of k and residue field of K^{nr}
G_K	the absolute Galois group $\text{Gal}(K^{\text{sep}}/K)$
I_K	inertia subgroup of G_K
Frob	a choice of (arithmetic) Frobenius element in G_K
$\bar{x}$ or $x \bmod \mathfrak{m}$	image in the residue field $\bar{k}$ for $x \in \bar{K}$ with $v(x) \geqslant 0$
C	hyperelliptic curve given by $y^2 = f(x)$
c	leading coefficient of $f(x)$
$\mathcal{R}$	set of roots of $f(x)$ in K^{sep}
g	genus of C
$\mathcal{C}_{\min}$	minimal regular model of $C/\mathcal{O}_{K^{\text{nr}}}$
$\mathcal{C}_{\bar{k}}^{\min}$	special fibre of $\mathcal{C}_{\min}$
Jac C	Jacobian of C

We will say C is semistable if C has semistable reduction. Similarly C is tame if C acquires semistable reduction over a tame extension of K. If $p > 2g+1$, C is always tame, see Remark 5.7.

3 | CLUSTERS

Definition 3.1 (Clusters and cluster pictures). A *cluster* is a non-empty subset $\mathfrak{s} \subseteq \mathcal{R}$ of the form $\mathfrak{s} = D \cap \mathcal{R}$ for some disc $D = \{x \in \bar{K} \mid v(x - z) \geqslant d\}$ for some $z \in \bar{K}$ and $d \in \mathbb{Q}$.

For a cluster $\mathfrak{s}$ with $|\mathfrak{s}| > 1$, its *depth* $d_{\mathfrak{s}}$ is the maximal d for which $\mathfrak{s}$ is cut out by such a disc, that is $d_{\mathfrak{s}} = \min_{r,r' \in \mathfrak{s}} v(r - r')$. If moreover $\mathfrak{s} \neq \mathcal{R}$, then its *relative depth* is $\delta_{\mathfrak{s}} = d_{\mathfrak{s}} - d_{P(\mathfrak{s})}$, where $P(\mathfrak{s})$ is the smallest cluster with $\mathfrak{s} \subsetneq P(\mathfrak{s})$ (the *parent* cluster).

We refer to this data as the *cluster picture* of C.

Remark 3.2. The Galois group acts on clusters via its action on the roots. It preserves depths and containments of clusters.

Notation 3.3. We draw cluster pictures by drawing roots $r \in \mathcal{R}$ as ●, and draw ovals around roots to represent clusters (of size > 1), such as

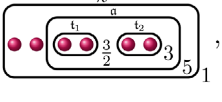
The subscript on the largest cluster $\mathcal{R}$ is its depth, while the subscripts on the other clusters are their relative depths.

Notation 3.4. For a cluster $\mathfrak{s}$ we use the following terminology.

size of $\mathfrak{s}$	$\lvert\mathfrak{s}\rvert$
$\mathfrak{s}'$ a child of $\mathfrak{s}$, $\mathfrak{s}' < \mathfrak{s}$	$\mathfrak{s}'$ is a maximal subcluster of $\mathfrak{s}$
parent of $\mathfrak{s}$, $P(\mathfrak{s})$	$P(\mathfrak{s})$ is the smallest cluster with $\mathfrak{s} \subsetneq P(\mathfrak{s})$
singleton	cluster of size 1
proper cluster	cluster of size > 1
even cluster	cluster of even size
odd cluster	cluster of odd size
übereven cluster	even cluster all of whose children are even
twin	cluster of size 2
cotwin	non-übereven cluster with a child of size $2g$
principal cluster $\mathfrak{s}$	if $\lvert\mathfrak{s}\rvert \neq 2g+2$: $\mathfrak{s}$ is proper, not a twin or a cotwin;
	if $\lvert\mathfrak{s}\rvert = 2g+2$: $\mathfrak{s}$ has $\geqslant 3$ children and is not a cotwin
$\mathfrak{s}^*$	if $\mathfrak{s}$ is not a cotwin:
	smallest $\mathfrak{s}^* \supseteq \mathfrak{s}$ that does not have an übereven parent;
	if $\mathfrak{s}$ is a cotwin: the child of $\mathfrak{s}$ of size $2g$
$\mathfrak{s} \wedge \mathfrak{s}'$	smallest cluster containing $\mathfrak{s}$ and $\mathfrak{s}'$
$\tilde{\mathfrak{s}}$	set of odd children of $\mathfrak{s}$
centre $z_{\mathfrak{s}}$	a choice of $z_{\mathfrak{s}} \in K^{\mathrm{sep}}$ with $\min_{r\in\mathfrak{s}} v(z_{\mathfrak{s}} - r) = d_{\mathfrak{s}}$
$\theta_{\mathfrak{s}}$	a choice of $\sqrt{c\prod_{r\notin\mathfrak{s}}(z_{\mathfrak{s}} - r)}$
$\epsilon_{\mathfrak{s}}$	$\epsilon_{\mathfrak{s}} : G_K \to \{\pm 1\}$, $\epsilon_{\mathfrak{s}}(\sigma) = \frac{\sigma(\theta_{\mathfrak{s}^*})}{\theta_{(\sigma\mathfrak{s})^*}} \bmod \mathfrak{m}$ if $\mathfrak{s}$ even or a cotwin,
	$\epsilon_{\mathfrak{s}} = 0$ otherwise
$\nu_{\mathfrak{s}}$	$= v(c) + \lvert\mathfrak{s}\rvert d_{\mathfrak{s}} + \sum_{r\notin\mathfrak{s}} d_{\{r\}\wedge\mathfrak{s}}$, for a proper cluster $\mathfrak{s}$
$\tilde{\lambda}_{\mathfrak{s}}$	$= \frac{1}{2}(v(c) + \lvert\tilde{\mathfrak{s}}\rvert d_{\mathfrak{s}} + \sum_{r\notin\mathfrak{s}} d_{\{r\}\wedge\mathfrak{s}})$, for a proper cluster $\mathfrak{s}$

Remark 3.5. For even clusters and cotwins, $\epsilon_{\mathfrak{s}}$ does not depend on the choice of centre of $\mathfrak{s}$. When restricted to the stabiliser of $\mathfrak{s}$, it is a homomorphism and does not depend on the choice of square root of $\theta_{\mathfrak{s}}^2$.

Example 3.6. Consider $C : y^2 = (x^2 + 7^2)(x^2 - 7^{15})(x - 7^6)(x - 7^6 - 7^9)$ over $\mathbb{Q}_7$. Its cluster picture is

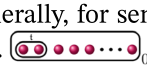


, with $\mathcal{R} = \{7i, -7i, 7^{\frac{15}{2}}, -7^{\frac{15}{2}}, 7^6, 7^6 + 7^9\}$, where $i^2 = -1$.

- *Depths and relative depths*: For each pair of roots r, r' in the picture, $v(r-r') \geqslant 1$, and $v(7i - 7^6) = 1$ so that $d_{\mathcal{R}} = 1$. Similarly $\mathfrak{a} = \{7^{\frac{15}{2}}, -7^{\frac{15}{2}}, 7^6, 7^6+7^9\}$ is a cluster of depth $d_{\mathfrak{a}} = 6$ and therefore relative depth $\delta_{\mathfrak{a}} = 5$. Finally, $\mathfrak{t}_1 = \{7^{\frac{15}{2}}, -7^{\frac{15}{2}}\}$ has depth $d_{\mathfrak{t}_1} = \frac{15}{2}$ and $\mathfrak{t}_2 = \{7^6, 7^6+7^9\}$ has depth $d_{\mathfrak{t}_2} = 9$. The only other clusters are singletons hence are not assigned any depth.
- *Children*: The children of $\mathcal{R}$ are $\{7i\}, \{-7i\}$ and $\mathfrak{a}$, so $\tilde{\mathcal{R}} = \{\{7i\},\{-7i\}\}$. The children of $\mathfrak{a}$ are $\mathfrak{t}_1$ and $\mathfrak{t}_2$, so $\tilde{\mathfrak{a}}$ is empty.
- *Types*: $\mathcal{R}, \mathfrak{a}, \mathfrak{t}_1, \mathfrak{t}_2$ are proper and even. The only odd clusters are singletons. Both $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are twins, $\mathfrak{a}$ is übereven and $\mathcal{R}$ is a cotwin. The only principal cluster is $\mathfrak{a}$.
- $\mathfrak{s}^*$ and $\mathfrak{s}_1 \wedge \mathfrak{s}_2$: $\mathfrak{t}_1^* = \mathfrak{t}_2^* = \mathfrak{a}^* = \mathcal{R}^* = \mathfrak{a}$, $\mathfrak{t}_1 \wedge \mathfrak{t}_2 = \mathfrak{a}$, $\mathfrak{t}_1 \wedge \mathfrak{a} = \mathfrak{a}$ and $\mathfrak{t}_1 \wedge \{7i\} = \mathcal{R}$.
- $z_{\mathfrak{s}}$ and $\epsilon_{\mathfrak{s}}$: Pick $z_{\mathcal{R}} = z_{\mathfrak{a}} = z_{\mathfrak{t}_1} = 0$ and $z_{\mathfrak{t}_2} = 7^6$. As $\mathfrak{t}_1^* = \mathfrak{t}_2^* = \mathfrak{a}^* = \mathcal{R}^* = \mathfrak{a}$, we get $\epsilon_{\mathfrak{t}_1} = \epsilon_{\mathfrak{t}_2} = \epsilon_{\mathcal{R}} = \epsilon_{\mathfrak{a}}$. With our choice of $z_{\mathfrak{a}}$ we obtain $\theta_{\mathfrak{a}} = \sqrt{(0-7i)(0+7i)} = \pm 7$. Say we choose $\theta_{\mathfrak{a}} = 7$, then for any $\sigma \in G_K$ we have $\epsilon_{\mathfrak{a}}(\sigma) = \frac{\sigma(\theta_{\mathfrak{a}})}{\theta_{\sigma\mathfrak{a}}} = +1$.

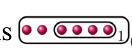
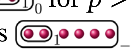
Example 3.7. Suppose $C/\mathbb{Q}_p : y^2 = f(x)$ with $f(x) \in \mathbb{Z}_p[x]$ monic. Suppose also that $f(x) \bmod p$ has at least two distinct roots, equivalently $d_{\mathcal{R}} = 0$. Consider the reduction $\bar{C}/\mathbb{F}_p : y^2 = \bar{f}(x)$.

(i) A child of $\mathcal{R}$ consists of roots that have the same image in the residue field. For example if $p = 5$ and $\mathcal{R} = \{0,1,2,3,5,8,13\}$, we have the cluster picture  and $\bar{C} : y^2 = x^2(x-1)(x-2)(x-3)^3$.

(ii) If $f(x) \bmod p$ has a double root and no other repeated roots, then the cluster picture has a twin $\mathfrak{t}$ and $\bar{C}$ has a node. Generally, for semistable curves, twins contribute nodes to the special fibre of the stable model. 

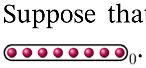
(iii) The normalisation of $\bar{C}$ is obtained by removing the maximal square factor in $\bar{f}(x)$, so the new roots are in 1:1 correspondence with the odd clusters. Explicitly, it is the hyperelliptic curve given by $y^2 = \prod_{\mathfrak{s}\in\tilde{\mathcal{R}}}(x - \bar{z}_{\mathfrak{s}})$. For example, for the curve in (i), the normalisation is given by $y^2 = (x-1)(x-2)(x-3)$.

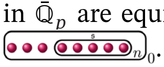
(iv) When $\mathcal{R}$ is übereven, the normalisation of $\bar{C}$ is $y^2 = 1$, which is a union of two lines. Generally, for semistable curves, übereven clusters contribute pairs of $\mathbb{P}^1$s to the special fibre of both semistable and regular models of $C/\mathbb{Q}_p^{\mathrm{nr}}$.

(v) Suppose that $\mathcal{R} = \{1, 2, p, 2p, 3p, 4p\}$ so the cluster picture is  for $p > 3$. Applying the change of variable $x' = \frac{1}{x}$ gives a curve whose cluster picture is . Generally, changing the model can convert twins to cotwins and vice versa, and the number of twins plus cotwins is model independent.

(vi) For a curve as in (ii), the node on $\bar{C}$ is split if and only if $\prod_{r\notin\mathfrak{t}}(\bar{z}_t - \bar{r})$ is a square in $\mathbb{F}_p$. Equivalently, if and only if $\epsilon_{\mathfrak{t}}(\mathrm{Frob}) = +1$. Generally, ϵ keeps track of whether the nodes are split or non-split and similar data.

Example 3.8. Let $C/\mathbb{Q}_p : y^2 = f(x)$ with $f(x) \in \mathbb{Z}_p[x]$, $\deg(f) = 8$ and $v(c) \geqslant 0$.

(i) Suppose that $f(x) \bmod p$ has distinct roots, equivalently that the cluster picture of C is . In this case, $\nu_{\mathcal{R}} = v(c)$. Here when $\nu_{\mathcal{R}}$ is even, C has good reduction and when $\nu_{\mathcal{R}}$ is odd it is a quadratic twist of such a curve.

(ii) Suppose that $f(x) \bmod p$ has a repeated root of multiplicity 5 and the corresponding roots in $\bar{\mathbb{Q}}_p$ are equidistant with distance $v(r_i - r_j) = n$, equivalently the cluster picture of C is .

The substitution $x' = \frac{x-z_{\mathfrak{s}}}{p^n}$ gives $f(x) = c\prod_{r\in\mathcal{R}}(p^n x' + z_{\mathfrak{s}} - r)$. Observe that $v(z_{\mathfrak{s}} - r) = d_{\mathfrak{s}} = n$ for $r \in \mathfrak{s}$ and $v(z_{\mathfrak{s}} - r) = d_{\{r\}\wedge\mathfrak{s}} = 0$ otherwise. The equation for C becomes

$$y^2 = cp^{5n}\prod_{r\notin\mathfrak{s}}(p^n x' + z_{\mathfrak{s}} - r)\prod_{r\in\mathfrak{s}}(x' + \frac{z_{\mathfrak{s}} - r}{p^n}).$$

Note that $v(cp^{5n})$ is precisely $v(c) + |\mathfrak{s}|d_{\mathfrak{s}} + \sum_{r\notin\mathfrak{s}} d_{\{r\}\wedge\mathfrak{s}} = \nu_{\mathfrak{s}}$, and by construction each factor of the above polynomial has integral coefficients.

In general, for any proper cluster $\mathfrak{s}$ the above change of variable will give an integral equation for C of the form $cp^m h(x)$ where $v(c) + m = \nu_{\mathfrak{s}}$ and $h(x)$ is integral. When $n \in \mathbb{Z}$, $z_{\mathfrak{s}} \in \mathbb{Q}_p$ and $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$, the substitution $y = y' p^{\frac{\nu_{\mathfrak{s}}}{2}}$ gives an equation for $C/\mathbb{Q}_p$ whose reduction is of the form

$$y^2 = (\text{constant})\prod_{r\in\mathfrak{s}}(x - r').$$

When $\mathfrak{s}$ is principal, this is a curve over $\mathbb{F}_p$ of genus at least 1.

Example 3.9. Consider the two curves $C_1 : y^2 = x^6 - p$ and $C_2 : y^2 = x(x^5 - p)$. These have cluster picture $_n$, where $n = \frac{1}{6}$ for C_1 and $n = \frac{1}{5}$ for C_2. These curves have $2\tilde{\lambda}_{\mathcal{R}} = v(c) + |\tilde{\mathcal{R}}|d_{\mathcal{R}} + \sum_{r\notin\mathcal{R}} d_{\{r\}\wedge\mathcal{R}} = 6n$. The denominator of $2\tilde{\lambda}_{\mathcal{R}}$ is either 1 or 5. This reflects the different inertia action on the roots: it has no fixed points for C_1 and one fixed point for C_2.

The general case is more subtle. Roughly, for a proper cluster $\mathfrak{s}$, the denominator of $\tilde{\lambda}_{\mathfrak{s}}$ is related to the inertia action on $\tilde{\mathfrak{s}}$ and to the inertia action by geometric automorphisms on the reduced curve associated to $\mathfrak{s}$ à la Example 3.8.

References. 3.1–3.4: [9, Section 1], [10, Section 3.3]. 3.5: [9, Remark 1.14]. 3.7(ii),(iv): [9, Theorem 8.5]. 3.7(v): [9, Theorem 14.4], [10, Proposition 5.24]. 3.9: [9, Section 8, Theorem 8.7(i)].

4 | BY TREES

Definition 4.1 (BY tree). A *BY tree* is a finite tree T with a *genus* function $g : V(T) \to \mathbb{Z}_{\geqslant 0}$ on vertices, a length function $\delta : E(T) \to \mathbb{R}_{>0}$ on edges, and a 2-colouring blue/yellow on vertices and edges such that

(1) yellow vertices have genus 0, degree $\geqslant 3$, and only yellow incident edges;
(2) blue vertices of genus 0 have at least one yellow incident edge;
(3) at every vertex, $2g(v) + 2 \geq \#$ blue incident edges at v.

Note that all leaves (vertices of degree 1) are necessarily blue.

Notation 4.2. In diagrams, yellow edges are drawn squiggly () and yellow vertices hollow () for the benefit of viewing them in black and white. We write the genus of a blue vertex inside the vertex (); we omit it for blue vertices with genus 0. We write the length of edges next to them.

Definition 4.3. The BY tree T_C associated to C is given by:

- one vertex $v_{\mathfrak{s}}$ for every proper cluster $\mathfrak{s}$, coloured yellow if $\mathfrak{s}$ is übereven and blue otherwise;
- for every pair $\mathfrak{s}' < \mathfrak{s}$ with $\mathfrak{s}'$ proper, link $v_{\mathfrak{s}'}$ and $v_{\mathfrak{s}}$ with an edge, yellow of length $2\delta_{\mathfrak{s}'}$ if $\mathfrak{s}'$ is even and blue of length $\delta_{\mathfrak{s}'}$ if $\mathfrak{s}'$ is odd;
- if $\mathcal{R}$ has size $2g+2$ and is a union of two proper children, remove $v_{\mathcal{R}}$ and merge the two remaining edges (adding their lengths);
- if $\mathcal{R}$ has size $2g+2$ and has a child $\mathfrak{s}$ of size $2g+1$, remove $v_{\mathcal{R}}$ and the edge between $v_{\mathcal{R}}$ and $v_{\mathfrak{s}}$;
- the genus $g(v_{\mathfrak{s}})$ of a blue vertex $v_{\mathfrak{s}}$ is defined so that $|\tilde{\mathfrak{s}}| = 2g(v_{\mathfrak{s}})+2$ or $2g(v_{\mathfrak{s}})+1$.

Definition 4.4. An *isomorphism of BY trees* $T \to T'$ is a pair (α, ϵ) where:

- α is a graph isomorphism $T \to T'$ that preserves edge lengths, genera of vertices and colours; and
- for every connected component Y of the yellow part $T_y \subset T$, $\epsilon(Y) \in \{\pm 1\}$.

Equivalently, ϵ is a collection of signs $\epsilon(v) \in \{\pm 1\}$ and $\epsilon(e) \in \{\pm 1\}$ for every yellow vertex and yellow edge, such that $\epsilon(v) = \epsilon(e)$ whenever e ends at v. Isomorphisms are composed by the cocycle rule

$$(\alpha, \epsilon_\alpha) \circ (\beta, \epsilon_\beta) = \big(\alpha \circ \beta, \bullet \mapsto \epsilon_\beta(\bullet)\epsilon_\alpha(\beta(\bullet))\big).$$

An *automorphism* of T is an isomorphism from T to itself.

Definition 4.5. The induced action of G_K is given by $\sigma \mapsto (\alpha_\sigma, \varepsilon_\sigma) \in \operatorname{Aut} T_C$ with $\alpha_\sigma(v_{\mathfrak{s}}) = v_{\sigma(\mathfrak{s})}$ for all vertices $v_{\mathfrak{s}}$, and $\epsilon_\sigma(Y) = \epsilon_{\mathfrak{s}_Y}(\sigma)$ for yellow components Y. Here the cluster $\mathfrak{s}_Y$ is taken so that $v_{\mathfrak{s}_Y}$ is any vertex in the closure of Y, other than the maximal one among these clusters. Note that $\epsilon_{\mathfrak{s}_Y}(\sigma)$ depends on the choices of square roots of θ^2.

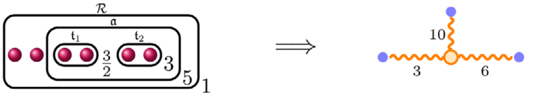
Notation 4.6. We draw arrows between edges and signs above yellow components to represent automorphisms.

Remark 4.7. For semistable curves, inertia maps to the identity in $\operatorname{Aut} T_C$, that is $\alpha_\sigma = \mathrm{id}$ and $\epsilon_\sigma(Y) = +1$ for all $\sigma \in I_K$ and all yellow components Y.

Lemma 4.8. *The genus of the curve satisfies*

$$g = \#(\textit{connected components of the blue part of } T_C) - 1 + \sum_{v \in V(T_C)} g(v).$$

Example 4.9. Consider the cluster picture from Example 3.6. There are four proper clusters $\mathcal{R}, \mathfrak{a}, \mathfrak{t}_1$ and $\mathfrak{t}_2$ so the BY tree has four vertices $v_{\mathcal{R}}, v_{\mathfrak{a}}, v_{\mathfrak{t}_1}, v_{\mathfrak{t}_2}$, where only $v_{\mathfrak{a}}$ is yellow since $\mathfrak{a}$ is übereven. There are three yellow edges corresponding to the three even children $\mathfrak{a} < \mathcal{R}, \mathfrak{t}_1 < \mathfrak{a}, \mathfrak{t}_2 < \mathfrak{a}$, of length $2 \times 5, 2 \times \frac{3}{2}, 2 \times 3$, respectively.



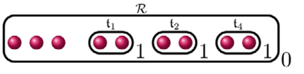
Remark 4.10.

(i) The depth $d_{\mathcal{R}}$ is not relevant for the BY tree.
(ii) The yellow part forms an open subset (since yellow vertices correspond to übereven clusters, which are even and only have even children).
(iii) One can reconstruct the cluster picture from the BY tree and $d_{\mathcal{R}}$, provided that there is a vertex $v_{\mathcal{R}}$ and it is identified.

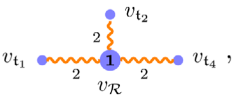
Example 4.11. Consider the curve $C/\mathbb{Q}_{11}$ given by $y^2 = f(x)$ with $f(x)$ monic with set of roots

$$\mathcal{R} = \{0, 1, 2, \ \zeta_7 - 11, \zeta_7 + 11, \ \zeta_7^2 - 11, \zeta_7^2 + 11, \ \zeta_7^4 - 11, \zeta_7^4 + 11\},$$

where $\zeta_7^7 = 1$ and $\zeta_7^3 + 5\zeta_7^2 + 4\zeta_7 + 10 \equiv 0 \bmod 11$. Its cluster picture and BY tree are



and



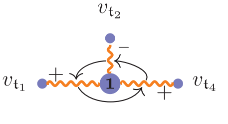
,

with centres for the twins $z_{\mathfrak{t}_1} = \zeta_7$, $z_{\mathfrak{t}_2} = \zeta_7^2$ and $z_{\mathfrak{t}_4} = \zeta_7^4$. Note that $\mathrm{Frob}(\mathfrak{t}_1) = \mathfrak{t}_4$, $\mathrm{Frob}(\mathfrak{t}_4) = \mathfrak{t}_2$ and $\mathrm{Frob}(\mathfrak{t}_2) = \mathfrak{t}_1$. We find that

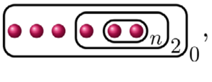
$$\theta_{\mathfrak{t}_1}^2 = (\zeta_7 - \zeta_7^2)^2(\zeta_7 - \zeta_7^4)^2\zeta_7(\zeta_7 - 1)(\zeta_7 - 2) \equiv \zeta_7^2 + 3\zeta_7 + 7 \bmod 11,$$

and similarly for $\theta_{\mathfrak{t}_2}^2$ and $\theta_{\mathfrak{t}_4}^2$. We can pick $\theta_{\mathfrak{t}_1}, \theta_{\mathfrak{t}_2}, \theta_{\mathfrak{t}_4}$ so that $\mathrm{Frob}(\theta_{\mathfrak{t}_1}) = \theta_{\mathfrak{t}_4}$, $\mathrm{Frob}(\theta_{\mathfrak{t}_4}) = \theta_{\mathfrak{t}_2}$ and therefore $\epsilon_{\mathfrak{t}_1}(\mathrm{Frob}) = \epsilon_{\mathfrak{t}_4}(\mathrm{Frob}) = +1$. Then $\epsilon_{\mathfrak{t}_2}(\mathrm{Frob}) \equiv \frac{\mathrm{Frob}^3(\theta_{\mathfrak{t}_1})}{\theta_{\mathfrak{t}_1}} \bmod 11$. One checks that $\theta_{\mathfrak{t}_1}^2$ is not a square in $\mathbb{F}_{11}(\zeta_7)$, so $\epsilon_{\mathfrak{t}_2}(\mathrm{Frob}) = -1$.

In terms of the BY tree, Frob permutes the three edges cyclicly. Here the yellow components are the three edges $v_{\mathcal{R}}v_{\mathfrak{t}_1}$, $v_{\mathcal{R}}v_{\mathfrak{t}_2}$, $v_{\mathcal{R}}v_{\mathfrak{t}_4}$, and $\epsilon_{\mathrm{Frob}}(v_{\mathcal{R}}v_{\mathfrak{t}_1}) = \epsilon_{\mathrm{Frob}}(v_{\mathcal{R}}v_{\mathfrak{t}_4}) = +1$, while $\epsilon_{\mathrm{Frob}}(v_{\mathcal{R}}v_{\mathfrak{t}_2}) = -1$.



Example 4.12. Let $C/\mathbb{Q}_p : y^2 = (x-1)(x-2)(x-3)(x-p^2)(x-p^{n+2})(x+p^{n+2})$ for $p \geqslant 5$ and $n \geqslant 1$. The substitutions $(x', y') = (\frac{1}{x-1}, \frac{y}{x-1})$ and $(x'', y'') = (\frac{1}{x}, \frac{y}{x})$ yield other models. Their cluster pictures are, respectively,



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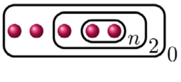
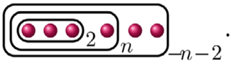


and



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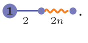
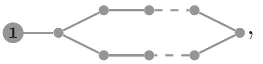
Note that these all have the same BY tree: 

. Generally, the BY tree is model independent.

Remark 4.13. For semistable curves the special fibre of the minimal regular model is the double cover of the BY tree ramified over the blue part (with all edge lengths halved). In Example 4.12

the dual graph is



where the loop has $2n$ vertices.

References. 4.3: [10, Table 5.3]. 4.5: [10, Table 4.20]. 4.7: Theorem 5.1. 4.8: [10, Definitions 3.23 and 3.33, Remark 3.24, Theorem 5.1]. 4.12: Theorem 8.3, Theorems 17.3 and 17.4.

5 | REDUCTION TYPE

In this section, we explain how to read off information about the reduction of both C and its Jacobian from the cluster picture of C.

Theorem 5.1 (Semistability criterion). *The curve C, or equivalently Jac C, is semistable if and only if the following three conditions are satisfied.*

1. *The field extension $K(\mathcal{R})/K$ given by adjoining the roots of $f(x)$ has ramification degree at most 2.*
2. *Every proper cluster is invariant under the action of the inertia group I_K.*
3. *Every principal cluster $\mathfrak{s}$ has $d_{\mathfrak{s}} \in \mathbb{Z}$ and $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$.*

Remark 5.2. It follows from Theorem 5.1 that C is semistable over any ramified quadratic extension of $K(\mathcal{R})$.

Theorem 5.3 (Good reduction of the curve). *The curve C has good reduction if and only if the following three conditions are all satisfied.*

1. *The field extension $K(\mathcal{R})/K$ is unramified.*
2. *Every proper cluster has size at least $2g+1$.*
3. *The (necessarily unique) principal cluster has $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$.*

Theorem 5.4 (Good reduction of the Jacobian). *The Jacobian of C has good reduction if and only if the following three conditions are all satisfied.*

1. *The field extension $K(\mathcal{R})/K$ is unramified.*
2. *Every cluster $\mathfrak{s} \neq \mathcal{R}$ is odd.*
3. *Every principal cluster $\mathfrak{s}$ has $\nu_{\mathfrak{s}} \in 2\mathbb{Z}$.*

A consequence of Theorems 5.3 and 5.4 is the following criterion for potentially good reduction.

Theorem 5.5 (Potentially good reduction of the curve or the Jacobian).

- *The curve C has potentially good reduction if and only if every proper cluster has size at least $2g+1$.*
- *The Jacobian, Jac C, has potentially good reduction if and only if every cluster $\mathfrak{s} \neq \mathcal{R}$ is odd.*

Theorem 5.6 (Potential toric rank of the Jacobian).

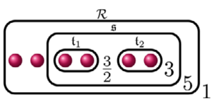
- *The potential toric rank of Jac C is equal to the number of even non-übereven clusters $\mathfrak{s} \neq \mathcal{R}$, less 1 if $\mathcal{R}$ is übereven.*
- *The Jacobian,* Jac C, *has potentially totally toric reduction if and only if every cluster has at most 2 odd children.*

Remark 5.7 (Tame reduction). The curve C, or equivalently Jac C, has tame reduction (semistable after tamely ramified extension) if and only if $K(\mathcal{R})/K$ is tamely ramified. In particular, this is always the case if $p > 2g + 1$ since then the wild inertia group acts trivially on the roots of the (degree $\leqslant 2g + 2$) polynomial $f(x)$.

Example 5.8. As in Example 3.6, we consider the genus 2 hyperelliptic curve

$$C : y^2 = (x^2 + 7^2)(x^2 - 7^{15})(x - 7^6)(x - 7^6 - 7^9)$$

over $\mathbb{Q}_7$ with cluster picture



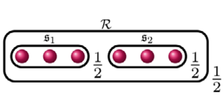
and $\mathcal{R} = \{7i, -7i, 7^{\frac{15}{2}}, -7^{\frac{15}{2}}, 7^6, 7^6 + 7^9\}$.

We have $d_{\mathcal{R}} = 1$. The single principal cluster $\mathfrak{s}$ has $d_{\mathfrak{s}} = 6$ and $|\mathfrak{s}| = 4$. We find:

- C is semistable. Indeed, $\mathbb{Q}_7(\mathcal{R}) = \mathbb{Q}_7(i, \sqrt{7})$ has ramification degree 2 over $\mathbb{Q}_7$. The inertia group swaps the roots $7^{\frac{15}{2}}$ and $-7^{\frac{15}{2}}$ which lie in a twin, and fixes all others, so that every proper cluster is fixed by inertia. Finally, $d_{\mathfrak{s}} \in \mathbb{Z}$ and $\nu_{\mathfrak{s}} = 4 \cdot d_{\mathfrak{s}} + 2d_{\mathcal{R}} = 26 \in 2\mathbb{Z}$;
- C does not have potentially good reduction since the cluster $\mathfrak{s}$ has size $4 < 2g + 1 = 5$. In fact, Jac C has totally toric reduction. Indeed, C is already semistable over $\mathbb{Q}_7$, and every cluster has at most 2 odd children ($\mathcal{R}$ and the twins $\mathfrak{t}_1$ and $\mathfrak{t}_2$ each have two odd children, whilst $\mathfrak{a}$ has no odd children).

Remark 5.9. Any hyperelliptic curve $C : y^2 = f(x)$ with the same cluster picture as the one in Example 5.8 (same depths, all proper clusters inertia invariant) and such that $f(x)$ has unit leading coefficient, is necessarily also semistable with totally toric reduction, by the same argument.

Example 5.10. Consider the genus 2 hyperelliptic curve $C : y^2 = x^6 - 27$ over $\mathbb{Q}_3$. Its cluster picture is



with $\mathcal{R} = \{\sqrt{3}, \zeta_3\sqrt{3}, \zeta_3^2\sqrt{3}, -\sqrt{3}, -\zeta_3\sqrt{3}, -\zeta_3^2\sqrt{3}\}$,

for a fixed primitive third root of unity ζ_3. The non-principal cluster $\mathcal{R}$ has depth $\frac{1}{2}$, whilst the principal clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$ each have depth 1. We find:

- C is not semistable since the action of inertia swaps $\mathfrak{s}_1$ and $\mathfrak{s}_2$;
- C does not have potentially good reduction, since $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are both proper clusters of size $< 2g + 1$. On the other hand, Jac C *does* have potentially good reduction since $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are odd;

- C has tame reduction since $\mathbb{Q}_3(\mathcal{R}) = \mathbb{Q}_3(\sqrt{3}, \zeta_3)$ has ramification degree 2 over $\mathbb{Q}_3$. In fact, the minimal degree extension over which C is semistable is 4, realised by any totally ramified extension of this degree. To see this, note that the inertia group acts on the proper clusters through its unique order 2 quotient, whilst for $i = 1, 2$, we have $d_{\mathfrak{s}_i} \in \mathbb{Z}$ and $\nu_{\mathfrak{s}_i} = 3 \cdot 1 + 3 \cdot d_{\mathcal{R}} = 9/2$, so that C satisfies the semistability criterion (Theorem 5.1) over some $F/\mathbb{Q}_3$ if and only if the ramification degree of this extension is divisible by 4.

References. 5.1: [9, Theorem 1.8, Theorem 7.1, Appendix C]. 5.3–5.7: [9, Theorem 1.8, Theorem 10.3]. Background on reduction types: [9, Section 2] and references therein.

6 | SPECIAL FIBRE (SEMISTABLE CASE)

In this section, assuming that C/K is *semistable* and that $\mathcal{R}$ *is principal*, we describe the special fibre of the minimal regular model of C over $\mathcal{O}_{K^{\mathrm{nr}}}$. The case where $\mathcal{R}$ is not principal is dealt with in [9, Section 8].

Definition 6.1 (Leading terms and reduction maps). For a principal cluster $\mathfrak{s}$, define $c_{\mathfrak{s}} \in \bar{k}^\times$ and $\mathrm{red}_{\mathfrak{s}} : z_{\mathfrak{s}} + \pi^{d_{\mathfrak{s}}}\mathcal{O}_{\bar{K}} \to \bar{k}$ by

$$c_{\mathfrak{s}} = \frac{c}{\pi^{v(c)}} \prod_{r \notin \mathfrak{s}} \frac{z_{\mathfrak{s}} - r}{\pi^{v(z_{\mathfrak{s}} - r)}} \bmod \mathfrak{m} \qquad \text{and} \qquad \mathrm{red}_{\mathfrak{s}}(t) = \frac{t - z_{\mathfrak{s}}}{\pi^{d_{\mathfrak{s}}}} \bmod \mathfrak{m}.$$

For any cluster $\mathfrak{s}' < \mathfrak{s}$, we define $\mathrm{red}_{\mathfrak{s}}(\mathfrak{s}')$ to be $\mathrm{red}_{\mathfrak{s}}(r)$ for any choice of $r \in \mathfrak{s}'$.

Theorem 6.2 (Components). *The special fibre $C_{\bar{k}}^{\min}$ contains connected components $\Gamma_{\mathfrak{s}}$ corresponding to principal clusters $\mathfrak{s}$, given by the equations*

$$\Gamma_{\mathfrak{s}} : Y^2 = c_{\mathfrak{s}} \prod_{\text{odd } \mathfrak{o} < \mathfrak{s}} \left(X - \mathrm{red}_{\mathfrak{s}}(\mathfrak{o})\right) \prod_{\substack{\text{twin } \mathfrak{t} < \mathfrak{s} \\ \delta_{\mathfrak{t}} = \frac{1}{2}}} \left(X - \mathrm{red}_{\mathfrak{s}}(\mathfrak{t})\right)^2.$$

This component is irreducible when $\mathfrak{s}$ is non-übereven but splits into a pair of irreducible components $\Gamma_{\mathfrak{s}}^+, \Gamma_{\mathfrak{s}}^-$ otherwise (we write $\Gamma_{\mathfrak{s}}^+ = \Gamma_{\mathfrak{s}}^- = \Gamma_{\mathfrak{s}}$ in the non-übereven case). These components are linked by chains of $\mathbb{P}^1$s as described in Theorem 6.3.

Theorem 6.3 (Links). *The chains of $\mathbb{P}^1$s linking the irreducible components of Theorem 6.2 arise in exactly one of the following four ways.*

If $\mathfrak{s}' < \mathfrak{s}$ with both clusters principal and $\mathfrak{s}'$ is odd, we have a chain containing $\frac{1}{2}\delta_{\mathfrak{s}'} - 1$ components, linking $\Gamma_{\mathfrak{s}}$ to $\Gamma_{\mathfrak{s}'}$. If $\mathfrak{s}' < \mathfrak{s}$ with both clusters principal and $\mathfrak{s}'$ even, we have two chains containing $\delta_{\mathfrak{s}'} - 1$ components each, one linking $\Gamma_{\mathfrak{s}}^+$ to $\Gamma_{\mathfrak{s}'}^+$ and the other $\Gamma_{\mathfrak{s}}^-$ to $\Gamma_{\mathfrak{s}'}^-$. If $\mathfrak{t} < \mathfrak{s}$ with $\mathfrak{s}$ principal and $\mathfrak{t}$ a twin, we have a chain containing $2\delta_{\mathfrak{t}} - 1$ components, linking $\Gamma_{\mathfrak{s}}^+$ to $\Gamma_{\mathfrak{s}}^-$.

Theorem 6.4 (Frobenius action). *The Frobenius element* Frob *acts by permutation on the components of $C_{\bar{k}}^{\min}$ by sending $\Gamma_{\mathfrak{s}}^{\pm}$ to $\Gamma_{\mathrm{Frob}(\mathfrak{s})}^{\pm\epsilon_{\mathfrak{s}}(\mathrm{Frob})}$.*

Remark 6.5. There are also formulae describing the Frobenius action on the linking chains. See Theorem 8.1, and for full details [9, Theorem 8.5].

Theorem 6.6 (Reduction maps). *For a principal cluster $\mathfrak{s} \neq \mathcal{R}$, the reduction of a point $(x, y) \in C(K^{\mathrm{nr}})$ lies on $\Gamma_{\mathfrak{s}}$ if and only if*

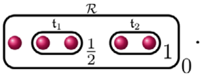
$$v(x - z_{\mathfrak{s}}) \geqslant d_{\mathfrak{s}} \text{ and } \mathrm{red}_{\mathfrak{s}}(x) \neq \mathrm{red}_{\mathfrak{s}}(\mathfrak{s}') \text{ for every proper } \mathfrak{s}' < \mathfrak{s}. \tag{6.7}$$

When these conditions are satisfied, the reduction is given by

$$(x, y) \mapsto \left(\mathrm{red}_{\mathfrak{s}}(x), \pi^{-\frac{\nu_{\mathfrak{s}}}{2}} y \cdot \prod_{\substack{\mathfrak{s}' < \mathfrak{s} \\ \delta_{\mathfrak{s}'} > \frac{1}{2}}} \left(\mathrm{red}_{\mathfrak{s}}(x) - \mathrm{red}_{\mathfrak{s}}(\mathfrak{s}')\right)^{-\left\lfloor \frac{|\mathfrak{s}'|}{2} \right\rfloor} \right). \tag{6.8}$$

If $\mathfrak{s} = \mathcal{R}$, then the reduction of $(x, y) \in C(K^{\mathrm{nr}})$ lies on $\Gamma_{\mathcal{R}}$ if and only if either (6.7) holds, or $v(x - z_{\mathcal{R}}) < d_{\mathfrak{s}}$. In the former case, the reduction is given by (6.8), whilst in the latter case (x, y) reduces to one of the points at infinity on $\Gamma_{\mathcal{R}}$.[†]

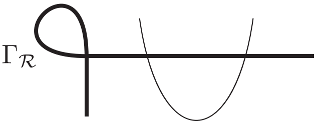
Example 6.9. Consider the genus 2 curve $C : y^2 = x((x + 1)^2 - 5)(x + 4)(x - 6)$ over $\mathbb{Q}_5$ with associated cluster picture



Picking $z_{\mathcal{R}} = 0$, we have $\mathrm{red}_{\mathcal{R}}(t) = t \bmod \mathfrak{m}$ and $c_{\mathcal{R}} = 1 \in \bar{\mathbb{F}}_5^{\times}$. The special fibre of the minimal regular model has a component coming from the unique principal cluster $\mathcal{R}$ given by the equation

$$\Gamma_{\mathcal{R}} : Y^2 = c_{\mathcal{R}} \cdot (X - \mathrm{red}_{\mathcal{R}}(0))(X - \mathrm{red}_{\mathcal{R}}(-4))^2 = X(X + 1)^2,$$

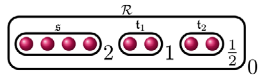
a genus 0 curve with a single node at $(X, Y) = (-1, 0)$. For the twin $\mathfrak{t}_1$, we have $2\delta_{\mathfrak{t}_1} - 1 = 0$ so that $\mathfrak{t}_1$ contributes no components (rather, it corresponds to the node on $\Gamma_{\mathcal{R}}$). On the other hand, the twin $\mathfrak{t}_2$ gives rise to a chain of $2\delta_{\mathfrak{t}_2} - 1 = 1$ projective lines from $\Gamma_{\mathcal{R}}$ to itself, as pictured below.



A point $(x, y) \in C(\mathbb{Q}_5^{\mathrm{nr}})$ reduces to a point on $\Gamma_{\mathcal{R}}$ if and only if either $x \notin \mathbb{Z}_5^{\mathrm{nr}}$, in which case it reduces to the unique point at infinity on $\Gamma_{\mathcal{R}}$, or $x \in \mathbb{Z}_5^{\mathrm{nr}}$ and $x \not\equiv \pm 1 \mod 5$. Since $\nu_{\mathcal{R}} = 0$, for points satisfying the second condition the reduction map is given by $(x, y) \mapsto (\bar{x}, \bar{y}(\bar{x} - 1)^{-1})$.

[†] When there are two points at infinity on $\Gamma_{\mathcal{R}}$ the reduction can be pinned down precisely by [9, Proposition 5.23 (i)].

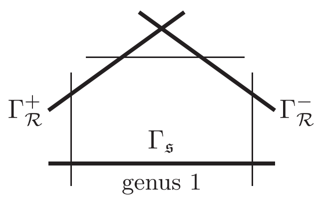
Example 6.10. Consider $C : y^2 = (x^4 - p^8)((x+1)^2 - p^2)((x-1)^2 - p)$ over $\mathbb{Q}_p$, with associated cluster picture



Then $\mathcal{R}$ and $\mathfrak{s}$ are the only principal clusters. Moreover, $\mathcal{R}$ is übereven. Taking $z_{\mathcal{R}} = z_{\mathfrak{s}} = 0$, we get associated components of $C^{\min}_{\bar{\mathbb{F}}_p}$:

$$\Gamma^+_{\mathcal{R}} : Y = X - 1, \quad \Gamma^-_{\mathcal{R}} : Y = 1 - X, \quad \text{and} \quad \Gamma_{\mathfrak{s}} : Y^2 = X^4 - 1.$$

The parent-child relation $\mathfrak{s} < \mathcal{R}$ gives rise to two chains of length $\delta_{\mathfrak{s}} = 1$, one linking $\Gamma^+_{\mathcal{R}}$ with $\Gamma_{\mathfrak{s}}$, and the other linking $\Gamma^-_{\mathcal{R}}$ with $\Gamma_{\mathfrak{s}}$. The twin $\mathfrak{t}_1$ gives rise to a chain of length $2\delta_{\mathfrak{t}_1} - 1 = 1$ linking $\Gamma^-_{\mathcal{R}}$ to $\Gamma^+_{\mathcal{R}}$. The twin $\mathfrak{t}_2$ has $2\delta_{\mathfrak{t}_2} - 1 = 0$ so contributes a chain of length 0 from $\Gamma^-_{\mathcal{R}}$ to $\Gamma^+_{\mathcal{R}}$, which is to be interpreted as a point of intersection between these two curves. The configuration of the components of the special fibre is shown below. Finally, since both $\mathcal{R}$ and $\mathfrak{s}$ are G_K-stable, and $\epsilon_{\mathcal{R}}(\sigma) = 1$ for all $\sigma \in G_K$, the Frobenius element fixes $\Gamma^+_{\mathcal{R}}$, $\Gamma^-_{\mathcal{R}}$, and $\Gamma_{\mathfrak{s}}$.



References. [9, Definition 8.4, Theorem 8.5].

7 | MINIMAL REGULAR MODEL (SEMISTABLE CASE)

Throughout this section, we assume that C is semistable. We also assume for simplicity that *all proper clusters have integral depth*, and that *there is no cluster* $\mathfrak{s} \neq \mathcal{R}$ of size $2g+1$.

Definition 7.1. An *integral disc* in $\bar{K}$ is a subset $D \subseteq \bar{K}$ of the form $D = D(z_D, d_D) = \{x \in \bar{K} : v(x - z_D) \geqslant d_D\}$ with $d_D \in \mathbb{Z}$. The point z_D is called a *centre* of D, and d_D is called its *depth*. The *parent disc* $P(D)$ of D is the disc with the same centre and depth $d_D - 1$. We also write $\nu_D(f) = v(c) + \sum_{r \in \mathcal{R}} \min\{d_D, v(r - z_D)\}$, and $\omega_D(f) \in \{0,1\}$ for the parity of $\nu_D(f)$.

We write $D(\mathcal{R})$ for the smallest disc containing $\mathcal{R}$. An integral disc D is called *valid* when $D \subseteq D(\mathcal{R})$ and $\#(\mathcal{R} \cap D) \geqslant 2$.

Construction of a regular model C^{disc} over $\mathcal{O}_{K^{\text{nr}}}$

Firstly, for each valid disc D, we let $f_D(x_D) \in \mathcal{O}_{K^{\text{nr}}}[x_D]$ denote the polynomial $f_D(x_D) = \pi^{-\nu_D(f)} f(\pi^{d_D} x_D + z_D)$. We set $\mathcal{U}_D$ to be the subscheme of $\mathbb{A}^2_{\mathcal{O}_{K^{\text{nr}}}}$ cut out by $y_D^2 = \pi^{\omega_D(f)} f_D(x_D)$.

We let $\mathcal{U}_D^\circ$ denote the open subscheme of $\mathcal{U}_D$ formed by removing all the points in the special fibre corresponding to repeated roots of the reduction of f_D (viewed as points on $\mathcal{U}_D$ with $y_D = 0$).

Next, for the maximal valid disc $D = D(\mathcal{R})$ we let $g_D(t_D) \in \mathcal{O}_{K^{\mathrm{nr}}}[t_D]$ denote the polynomial $g_D(t_D) = t_D^{\deg(f)} f_D(1/t_D)$. We set $\mathcal{W}_D$ to be the subscheme of $\mathbb{A}^2_{\mathcal{O}_{K^{\mathrm{nr}}}}$ cut out by $w_D^2 = \pi^{\omega_D(f)} g_D(t_D)$ if $\deg(f)$ is even, and $w_D^2 = \pi^{\omega_D(f)} t_D g_D(t_D)$ if $\deg(f)$ is odd. Again, we let $\mathcal{W}_D^\circ$ denote the open subscheme of $\mathcal{W}_D$ formed by removing all the points in the special fibre corresponding to repeated roots of the reduction of g_D (viewed as points on $\mathcal{W}_D$ with $w_D = 0$).

Finally, for each valid disc $D \neq D(\mathcal{R})$, we let $g_D(s_D, t_D) \in \mathcal{O}_{K^{\mathrm{nr}}}[s_D, t_D]/(s_D t_D - \pi)$ be the polynomial satisfying $g_D(\pi/t_D, t_D) = t_D^{\nu_D(f) - \nu_{P(D)}(f)} f_D(1/t_D)$ in $K^{\mathrm{nr}}(t_D)$. We set $\mathcal{W}_D$ to be the subscheme of $\mathbb{A}^3_{\mathcal{O}_{K^{\mathrm{nr}}}}$ cut out by the equations $s_D t_D = \pi$ and $w_D^2 = s_D^{\omega_D(f)} t_D^{\omega_{P(D)}(f)} g_D(s_D, t_D)$. Again, we let $\mathcal{W}_D^\circ$ denote the open subscheme of $\mathcal{W}_D$ formed by removing all the points in the special fibre corresponding to repeated roots of the reduction of g_D (viewed as points on $\mathcal{W}_D$ with $w_D = 0$).

Remark 7.2. An explicit formula for g_D is given in [9, Definition 3.15].

Theorem 7.3. *A regular model $\mathcal{C}^{\mathrm{disc}}$ of C over $\mathcal{O}_{K^{\mathrm{nr}}}$ is given by gluing each $\mathcal{W}_D^\circ$ to $\mathcal{U}_D^\circ$ for each valid D, and to $\mathcal{U}_{P(D)}^\circ$ for all valid $D \neq D(\mathcal{R})$ via the identifications*

$$t_D = 1/x_D = \pi/(x_{P(D)} - \pi^{1-d_D}(z_D - z_{P(D)})),$$

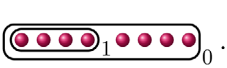
$$s_D = \pi x_D = x_{P(D)} - \pi^{1-d_D}(z_D - z_{P(D)}),$$

$$w_D = t_D^{\lfloor \nu_D(f)/2 \rfloor - \lfloor \nu_{P(D)}(f)/2 \rfloor} y_D = s_D^{\lfloor \nu_{P(D)}(f)/2 \rfloor - \lfloor \nu_D(f)/2 \rfloor} y_{P(D)}.$$

Remark 7.4. The regular model $\mathcal{C}^{\mathrm{disc}}$ above is not minimal in general: discs with $\omega_D(f) = 1$ produce $\mathbb{P}^1$s in the special fibre with multiplicity 2 and self-intersection -1. Blowing down these components yields the minimal regular model.

Remark 7.5. In the construction of $\mathcal{C}^{\mathrm{disc}}$ in [9, Proposition 5.5] for general semistable C, the scheme $\mathcal{U}_D^\circ$ is defined by removing from the special fibre of $\mathcal{U}_D$ all points corresponding to the maximal valid subdiscs of D. Under our extra assumptions, this is equivalent to the reduction of f_D having a repeated root at this point. This is untrue when C has a twin of half-integral depth; see Example 7.7.

Example 7.6. Consider $C : y^2 = (x^4 - p^4)(x^4 - 1)$ over $\mathbb{Q}_p$. Its cluster picture is



.

Here, there are two valid discs $D = D(0, 0)$ and $D' = D(0, 1)$. These correspond to the two proper clusters in the cluster picture. Using $\nu_D(f) = 0$ and $\nu_{D'}(f) = 4$, we find

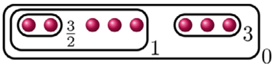
$$\mathcal{U}_D = \mathrm{Spec}\left(\frac{\mathbb{Z}_p^{\mathrm{nr}}[x, y]}{(y^2 - (x^4 - p^4)(x^4 - 1))}\right), \quad \mathcal{W}_D = \mathrm{Spec}\left(\frac{\mathbb{Z}_p^{\mathrm{nr}}[t, w]}{(w^2 - (1 - p^4 t^4)(1 - t^4))}\right)$$

and

$$\mathcal{U}_{D'} = \mathrm{Spec}\left(\frac{\mathbb{Z}_p^{\mathrm{nr}}[x',y']}{(y'^2-(x'^4-1)(p^4x'^4-1))}\right),\ \mathcal{W}_{D'} = \mathrm{Spec}\left(\frac{\mathbb{Z}_p^{\mathrm{nr}}[s',t',w']}{(s't'-p,w'^2-(1-t'^4)(s'^4-1))}\right).$$

We have $\mathcal{U}_D^\circ = \mathcal{U}_D \setminus \{(x,y,p)\}$, whereas $\mathcal{U}_{D'}^\circ = \mathcal{U}_{D'}$, $\mathcal{W}_D^\circ = \mathcal{W}_D$ and $\mathcal{W}_{D'}^\circ = \mathcal{W}_{D'}$. Using the identifications $t' = 1/x' = p/x$, $s' = px' = x$, $y' = y/p^2$ and $w' = t'^2y'$, we see that the special fibre of C^{disc} consists of two genus 1 curves which intersect in two points.

Example 7.7. Consider $C : y^2 = p(x^2-p^5)(x^3-p^3)((x-1)^3-p^9)$ over $\mathbb{Q}_p$ for $p \geqslant 5$. Its cluster picture is



.

There are six valid discs: $D(0,0)$, $D(0,1)$, $D(0,2)$, $D(1,1)$, $D(1,2)$, $D(1,3)$. Not all of these discs are minimal defining discs for clusters. For example, the cluster of relative depth 3 is cut out by three different valid discs.

Note that C has a proper cluster of non-integral depth, so Theorem 7.3 does not apply verbatim; we need the more general version from Remark 7.5. We give a few illustrative charts of the model C^{disc}.

For $D = D(0,0)$, we find

$$\mathcal{U}_D = \mathrm{Spec}\left(\frac{\mathbb{Z}_p^{\mathrm{nr}}[x,y]}{(y^2-p(x^2-p^5)(x^3-p^3)((x-1)^3-p^9))}\right)$$

and $\mathcal{U}_D^\circ = \mathcal{U}_D \setminus \{(x,y,p),(x-1,y,p)\}$. Note that the special fibre of $\mathcal{U}_D$ is non-reduced. More precisely its closure is a projective line of multiplicity 2 with self-intersection -1 and is blown down when constructing the minimal regular model (see Remark 7.4). The same applies to the component corresponding to $D(1,2)$.

For the disc $D_1 = D(1,1)$, we get

$$\mathcal{W}_{D_1} = \mathrm{Spec}\left(\frac{\mathbb{Z}_p^{\mathrm{nr}}[s_1,t_1,w_1]}{(s_1t_1-p,w_1^2-t_1(1-p^6t_1^3)((s_1+1)^2-p^5)((s_1+1)^3-p^3))}\right)$$

and $\mathcal{W}_{D_1}^\circ = \mathcal{W}_{D_1} \setminus \{(s_1+1,w_1,p)\}$. Similarly, for $D_1' = D(0,1)$, we have $\mathcal{W}_{D_1'}^\circ = \mathcal{W}_{D_1'} \setminus \{(s_1'-1,w_1',p)\}$.

Let $D_2 = D(0,2)$, then

$$\mathcal{U}_{D_2} = \mathrm{Spec}\left(\frac{\mathbb{Z}_p^{\mathrm{nr}}[x_2,y_2]}{(y_2^2-(x_2^2-p)(p^3x_2^3-1)((p^2x_2-1)^3-p^9))}\right).$$

Note that although the reduction of f_{D_2} has a double root at $x_2 = 0$, this double root does not correspond to a valid subdisc of D_2. Hence we do not remove this point in forming $\mathcal{U}_{D_2}^\circ$, so $\mathcal{U}_{D_2}^\circ = \mathcal{U}_{D_2}$ in this case.

References. 7.1: [9, Definition 4.4]. 7.3, 7.5: [9, Proposition 5.5]. 7.4: [9, Theorem 5.16].

8 | DUAL GRAPH OF SPECIAL FIBRE AND ITS HOMOLOGY (SEMISTABLE CASE)

In this section, C is semistable. Let $\mathcal{C}_{\min}$ be its minimal regular model over $\mathcal{O}_{K^{\mathrm{nr}}}$. The *dual graph* Υ_C consists of a vertex v_Γ for every irreducible component Γ of the geometric special fibre $\mathcal{C}^{\min}_{\bar{k}}$, with an edge connecting v_Γ and $v_{\Gamma'}$ for each intersection point of Γ and Γ' (self-intersections of Γ correspond to loops based at v_Γ). The action of Frob on $\mathcal{C}^{\min}_{\bar{k}}$ induces a corresponding action on Υ_C.

Theorem 8.1. *Υ_C consists of one vertex $v_{\mathfrak{s}}$ for every non-übereven principal cluster $\mathfrak{s}$ and two vertices $v^+_{\mathfrak{s}}$, $v^-_{\mathfrak{s}}$ for each übereven principal cluster $\mathfrak{s}$, connected by chains of edges as follows:*

Name	Endpoints		Length	Conditions
$L_{\mathfrak{s}'}$	$v_{\mathfrak{s}'}$	$v_{\mathfrak{s}}$	$\frac{1}{2}\delta_{\mathfrak{s}'}$	$\mathfrak{s}' < \mathfrak{s}$ both principal, $\mathfrak{s}'$ odd
$L^\pm_{\mathfrak{s}'}$	$v^\pm_{\mathfrak{s}'}$	$v^\pm_{\mathfrak{s}}$	$\delta_{\mathfrak{s}'}$	$\mathfrak{s}' < \mathfrak{s}$ both principal, $\mathfrak{s}'$ even
$L_{\mathfrak{t}}$	$v^-_{\mathfrak{s}}$	$v^+_{\mathfrak{s}}$	$2\delta_{\mathfrak{t}}$	$\mathfrak{s}$ principal, $\mathfrak{t} < \mathfrak{s}$ twin
$L_{\mathfrak{t}}$	$v^-_{\mathfrak{s}}$	$v^+_{\mathfrak{s}}$	$2\delta_{\mathfrak{t}}$	$\mathfrak{s}$ principal, $\mathfrak{s} < \mathfrak{t}$ cotwin
and, if $\mathcal{R}$ is non-principal				
$L_{\mathfrak{s}_1,\mathfrak{s}_2}$	$v_{\mathfrak{s}_1}$	$v_{\mathfrak{s}_2}$	$\frac{1}{2}(\delta_{\mathfrak{s}_1}+\delta_{\mathfrak{s}_2})$	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ with $\mathfrak{s}_1, \mathfrak{s}_2$ principal odd
$L^\pm_{\mathfrak{s}_1,\mathfrak{s}_2}$	$v^\pm_{\mathfrak{s}_1}$	$v^\pm_{\mathfrak{s}_2}$	$\delta_{\mathfrak{s}_1}+\delta_{\mathfrak{s}_2}$	$\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$ with $\mathfrak{s}_1, \mathfrak{s}_2$ principal even
$L_{\mathfrak{t}}$	$v^-_{\mathfrak{s}}$	$v^+_{\mathfrak{s}}$	$2(\delta_{\mathfrak{s}}+\delta_{\mathfrak{t}})$	$\mathcal{R} = \mathfrak{s} \sqcup \mathfrak{t}$ with $\mathfrak{s}$ principal even, $\mathfrak{t}$ twin

Here, we adopt the convention that $v^+_{\mathfrak{s}} = v^-_{\mathfrak{s}} = v_{\mathfrak{s}}$ if $\mathfrak{s}$ is not übereven, so, for example, if $\mathfrak{s}' < \mathfrak{s}$ are even non-übereven principal clusters, then there are two chains of edges $L^+_{\mathfrak{s}'}$, $L^-_{\mathfrak{s}'}$ connecting $v_{\mathfrak{s}'}$ and $v_{\mathfrak{s}}$.

Frobenius acts on Υ_C by $\mathrm{Frob}(v^\pm_{\mathfrak{s}}) = v^{\pm\epsilon_{\mathfrak{s}}(\mathrm{Frob})}_{\mathrm{Frob}(\mathfrak{s})}$, $\mathrm{Frob}(L^\pm_{\mathfrak{s}'}) = L^{\pm\epsilon_{\mathfrak{s}}(\mathrm{Frob})}_{\mathrm{Frob}(\mathfrak{s}')}$ and $\mathrm{Frob}(L_{\mathfrak{t}}) = \epsilon_{\mathfrak{t}}(\mathrm{Frob})L_{\mathrm{Frob}(\mathfrak{t})}$, where $-L$ denotes L with the opposite orientation.

The homology $H_1(\Upsilon_C, \mathbb{Z})$ is a finite-rank free $\mathbb{Z}$-module, carrying an induced Frobenius action and a *length pairing* $\langle\cdot,\cdot\rangle : H_1(\Upsilon_C,\mathbb{Z}) \otimes H_1(\Upsilon_C,\mathbb{Z}) \to \mathbb{Z}$ where $\langle\gamma_1,\gamma_2\rangle$ is the length of the intersection of cycles γ_1 and γ_2, interpreted in a suitably signed manner. The rank of $H_1(\Upsilon_C,\mathbb{Z})$ is the potential toric rank of Jac C, and the cokernel of the map $H_1(\Upsilon_C,\mathbb{Z}) \to H^1(\Upsilon_C,\mathbb{Z})$ induced by the length pairing is Frobenius-equivariantly isomorphic to the group of geometric components of the special fibre of the Néron model of Jac C.

Theorem 8.2. *Let A be the set of even non-übereven clusters except for $\mathcal{R}$.*

(1) *If $\mathcal{R}$ is not übereven, then $H_1(\Upsilon_C,\mathbb{Z}) = \mathbb{Z}[A]$ is the free $\mathbb{Z}$-module generated by symbols $\ell_{\mathfrak{s}}$ for $\mathfrak{s} \in A$.*
(2) *If $\mathcal{R}$ is übereven, let B be the set of those clusters $\mathfrak{s} \in A$ such that $\mathfrak{s}^* = \mathcal{R}$. Then $H_1(\Upsilon_C,\mathbb{Z}) \leqslant \mathbb{Z}[A]$ is the corank 1 submodule of $\mathbb{Z}[A]$ consisting of those elements $\sum_{\mathfrak{s}\in A} a_{\mathfrak{s}}\ell_{\mathfrak{s}}$ such that $\sum_{\mathfrak{s}\in B} a_{\mathfrak{s}} = 0$.*

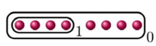
In both cases, Frobenius acts on $H_1(\Upsilon_C, \mathbb{Z})$ *is by* $\mathrm{Frob}(\ell_{\mathfrak{s}}) = \epsilon_{\mathfrak{s}}(\mathrm{Frob})\ell_{\mathrm{Frob}(\mathfrak{s})}$, *and the length pairing by*

$$\langle \ell_{\mathfrak{s}_1}, \ell_{\mathfrak{s}_2} \rangle = \begin{cases} 0 & \text{if } \mathfrak{s}_1^* \neq \mathfrak{s}_2^*, \\ 2(d_{\mathfrak{s}_1 \wedge \mathfrak{s}_2} - d_{P(\mathfrak{s}_1^*)}) & \text{if } \mathfrak{s}_1^* = \mathfrak{s}_2^* \neq \mathcal{R}, \\ 2(d_{\mathfrak{s}_1 \wedge \mathfrak{s}_2} - d_{\mathcal{R}}) & \text{if } \mathfrak{s}_1^* = \mathfrak{s}_2^* = \mathcal{R}. \end{cases}$$

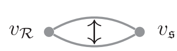
Theorem 8.3. Υ_C *is a double cover of* T_C *ramified over the blue part, the quotient map being induced by the hyperelliptic involution* ι. *Giving edges on* Υ_C *length 2 makes the identification* $\Upsilon_C/\langle\iota\rangle = T_C$ *distance preserving. The pre-image of a vertex* v *in* T_C *of genus* $g(v) > 0$ *is a vertex in* Υ_C *corresponding to a component of genus* $g(v)$ *in the special fibre.*

Example 8.4. Consider C over $\mathbb{Q}_p$ given by the equation

$$y^2 = x(x-p)(x-2p)(x-3p)(x-1)(x-2)(x-3)(x-4)$$

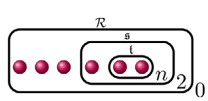
for $p \geqslant 5$. Its cluster picture is . Write $\mathfrak{s}$ for the cluster of size 4. According to Theorem 8.1, the dual graph Υ_C consists of two vertices $v_{\mathfrak{s}}$ and $v_{\mathcal{R}}$, connected by two edges $L_{\mathfrak{s}}^{\pm}$. The action of Frobenius on Υ_C fixes the two vertices, and acts on edges via $\mathrm{Frob}(L_{\mathfrak{s}}^{\pm}) = L_{\mathfrak{s}}^{\pm(\frac{6}{p})}$ where $(\frac{6}{p})$ is the Legendre symbol. In other words, the action on Υ_C is trivial if $p \equiv \pm 1$ or ± 5 mod 24, and interchanges the two edges $L_{\mathfrak{s}}^{+}$ and $L_{\mathfrak{s}}^{-}$ if $p \equiv \pm 7$ or ± 11 mod 24. In particular, the Frobenius action on Υ_C can be non-trivial even when the action on $\mathcal{R}$ is trivial. Pictorially, Υ_C is

or



From this, we see that $H_1(\Upsilon_C, \mathbb{Z}) = \mathbb{Z}$, the induced action of Frobenius is multiplication by $(\frac{6}{p})$, and the length pairing is $\langle m, n\rangle = 2mn$. This agrees with Theorem 8.2.

Example 8.5. Consider $C: y^2 = (x-1)(x-2)(x-3)(x-p^2)(x-p^{n+2})(x+p^{n+2})$ over $\mathbb{Q}_p$ for $p \geqslant 5$ (cf. Example 4.12). Its cluster picture is



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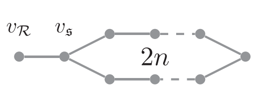
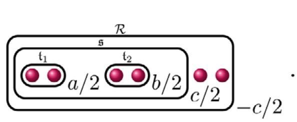
According to Theorem 8.1, the dual graph Υ_C consists of two vertices $v_{\mathcal{R}}$ and $v_{\mathfrak{s}}$, connected by a single edge $L_{\mathfrak{s}}$ and with a loop $L_{\mathfrak{t}}$ of $2n$ edges connecting $v_{\mathfrak{s}}$ to itself. Pictorially, Υ_C is



Example 8.6. Consider $C: y^2 = (x^2 - p^a)((x-1)^2 - p^b)(p^c x^2 - 1)$ over $\mathbb{Q}_p$, for some positive integers a, b, c. Its cluster picture is



.

We compute the homology $H_1(\Upsilon_C, \mathbb{Z})$ using Theorem 8.2, without first computing Υ_C. Except for $\mathcal{R}$ the even non-übereven clusters are the two twins $\mathfrak{t}_1$ and $\mathfrak{t}_2$, so $H_1(\Upsilon_C, \mathbb{Z})$ is free of rank 2, generated by $\ell_{\mathfrak{t}_1}$ and $\ell_{\mathfrak{t}_2}$. Frobenius acts on $H_1(\Upsilon_C, \mathbb{Z})$ by multiplication by $(\frac{-1}{p})$, and the length pairing has matrix $M = \begin{pmatrix} a+c & c \\ c & b+c \end{pmatrix}$.

From this, we see that the potential toric rank of Jac C is 2 (potentially totally toric reduction), and that the group of geometric components of the special fibre of the Néron model of Jac C has size $\det(M) = ab + bc + ca$. By computing the Smith normal form of M, we find that the group structure is $\mathbb{Z}/A\mathbb{Z} \oplus \mathbb{Z}/B\mathbb{Z}$, with $A = \gcd(a, b, c)$ and $B = (ab + bc + ca)/\gcd(a, b, c)$.

References. 8.1: [9, Theorem 8.5]. 8.2: [9, Theorem 1.14]. 8.3: [9, Theorem 5.18, Definitions D.6, D.9]. 8.6: Theorem 5.6, [9, Lemma 2.22].

9 | SPECIAL FIBRE OF THE MINIMAL REGULAR SNC MODEL (TAME CASE)

Assume C has *tame reduction*. We give a qualitative description of the special fibre of the minimal regular model of C with strict normal crossings (SNC), over $\mathcal{O}_{K^{\mathrm{nr}}}$. Denote this model $\mathcal{C}^{snc}$, special fibre $\mathcal{C}^{snc}_{\bar{k}}$. We assume $\mathcal{R}$ *is principal*.†

Notation 9.1. Let X be an I_K-orbit of clusters with $\mathfrak{s} \in X$. We say that X is proper/principal/odd/even/übereven/twin/singleton if $\mathfrak{s}$ is. If X' is another orbit, write $X' < X$ if $\mathfrak{s}' < \mathfrak{s}$ for some $\mathfrak{s}' \in X'$, and call X' *stable* if $|X'| = |X|$. Write $\mathfrak{s}_{\mathrm{sing}}$ for the set of size 1 children of $\mathfrak{s}$. Define $g_{\mathrm{ss}}(X) = 0$ if X is übereven, and so that $|\tilde{\mathfrak{s}}| \in \{2g_{\mathrm{ss}}(X) + 2, 2g_{\mathrm{ss}}(X) + 1\}$ otherwise. For X (henceforth) proper, write $d_X = d_{\mathfrak{s}}$, $\delta_X = \delta_{\mathfrak{s}}$ (for $\mathfrak{s} \neq \mathcal{R}$), $\lambda_X = \tilde{\lambda}_{\mathfrak{s}}$, and for X even $\epsilon_X = (-1)^{|X|(\nu_{\mathfrak{s}^*} - |\mathfrak{s}^*| d_{\mathfrak{s}^*})} \in \{\pm 1\}$.‡ Let $e_X \in \mathbb{Z}_{\geqslant 1}$ be minimal with $e_X|X|d_{\mathfrak{s}} \in \mathbb{Z}$ and $e_X|X|\nu_{\mathfrak{s}} \in 2\mathbb{Z}$. Write $d_X = \frac{a_X}{b_X}$ in lowest terms, and set $b'_X = b_X/\gcd(|X|, b_X)$. Finally, define $g(X)$ as $\lfloor g_{\mathrm{ss}}(X)/b'_X \rfloor$ if $|X|\lambda_X \in \mathbb{Z}$, $\lfloor g_{\mathrm{ss}}(X)/b'_X + 1/2 \rfloor$ if $|X|\lambda_X \notin \mathbb{Z}$ and b'_X is even, and 0 otherwise.

Definition 9.2. Let $t_1, t_2 \in \mathbb{Q}$ and $\mu \in \mathbb{N}$. Let n be minimal such that there exist coprime pairs $m_i, d_i \in \mathbb{Z}$ with $\mu t_1 = \frac{m_0}{d_0} > \frac{m_1}{d_1} > \dots > \frac{m_{n+1}}{d_{n+1}} = \mu t_2$ and with $m_i d_{i+1} - m_{i+1} d_i = 1$ for each $0 \leqslant i \leqslant n$. A *sloped chain of rational curves with parameters* (t_2, t_1, μ) is a chain of $\mathbb{P}^1$s $E_1, \dots, E_n$ with multiplicities μd_i, intersecting transversally. A *crossed tail* is a sloped chain with $\mu \in 2\mathbb{N}$ and two additional (disjoint) $\mathbb{P}^1$s of multiplicity $\mu/2$ intersecting E_n transversally.

Theorem 9.3. *Each principal I_K-orbit X of clusters gives rise to two 'central' components $\Gamma^{\pm}_X$ of $\mathcal{C}^{snc}_{\bar{k}}$ if X is übereven and $\epsilon_X = 1$, and one central component Γ_X ($= \Gamma^+_X = \Gamma^-_X$) otherwise. These have genus $g(X)$, and multiplicity $|X|e_X$ unless X is übereven with $\epsilon_X = -1$ when they have multiplicity $2|X|e_X$. These components are linked by (one or two) sloped chains of rational curves with parameters (t_2, t_1, μ) indexed by pairs $X' < X$ with X principal as follows:*

† This only serves to simplify the statements, see the references given for the general case.

‡ Let $I_{\mathfrak{s}}$ be the stabiliser of $\mathfrak{s}$ inside the inertia group I_K. Then the restriction of $\epsilon_{\mathfrak{s}}$ to $I_{\mathfrak{s}}$ is a character $I_{\mathfrak{s}} \to \{\pm 1\}$, and $\epsilon_X = -1$ if and only if this character is non-trivial.

Name	From	To	t_1	t_2	μ	Condition
$L_{X,X'}$	Γ_X	$\Gamma_{X'}$	$-\lambda_X$	$-\lambda_X - \frac{1}{2}\delta_{X'}$	$\lvert X'\rvert$	X' odd principal
$L^+_{X,X'}$	Γ^+_X	$\Gamma^+_{X'}$	$-d_X$	$-d_{X'}$	$\lvert X'\rvert$	X' even principal, $\epsilon_{X'} = 1$
$L^-_{X,X'}$	Γ^-_X	$\Gamma^-_{X'}$	$-d_X$	$-d_{X'}$	$\lvert X'\rvert$	X' even principal, $\epsilon_{X'} = 1$
$L_{X,X'}$	Γ_X	$\Gamma_{X'}$	$-d_X$	$-d_{X'}$	$2\lvert X'\rvert$	X' even principal, $\epsilon_{X'} = -1$
$L_{X'}$	Γ^-_X	Γ^+_X	$-d_X$	$-d_X - 2\delta_{X'}$	$\lvert X'\rvert$	X' twin, $\epsilon_{X'} = 1$

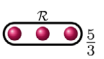
The central components Γ_X with $e_X > 1$ are intersected by (the first curve of one or more) sloped chains with parameters $(\frac{1}{\mu}\lfloor \mu t_1 - 1\rfloor, t_1, \mu)$ as follows:

From	No.	t_1	μ	Condition	
$\Gamma_\mathcal{R}$	1	$(g+1)d_\mathcal{R} - \lambda_\mathcal{R}$	1	$\lvert\mathcal{R}\rvert = 2g+1$	
$\Gamma^\pm_\mathcal{R}$	2	$-d_\mathcal{R}$	1	$\lvert\mathcal{R}\rvert = 2g+2, \epsilon_\mathcal{R} = 1$	
$\Gamma_\mathcal{R}$	1	$-d_\mathcal{R}$	2	$\lvert\mathcal{R}\rvert = 2g+2, e_\mathcal{R} > 2,\ \epsilon_\mathcal{R} = -1$	
Γ_X	$\frac{\lvert X\rvert\lvert\mathfrak{s}_{\text{sing}}\rvert}{b_X}$	$-\lambda_X$	b_X	$e_X > b_X/\lvert X\rvert$, $\lvert\mathfrak{s}_{\text{sing}}\rvert \geqslant 2\ \ \forall\, \mathfrak{s} \in X$	
Γ_X	1	$-d_X$	$2\lvert X\rvert$	No $X' < X$ is stable, and either	$\lambda_X \notin \mathbb{Z}$, $e_X > 2$
$\Gamma^\pm_X$	2	$-d_X$	$\lvert X\rvert$	X übereven or $g_{\text{ss}}(X) > 0$	$\lambda_X \in \mathbb{Z}$
Γ_X	1	$-\lambda_X$	$\lvert X\rvert$	X is not übereven, no odd proper $X' < X$ is stable,	
				and $g_{\text{ss}}(X) = 0$ or some singleton $X' < X$ is stable	

Finally (regardless of whether $e_X > 1$ or not), each Γ_X is intersected by the (first curve of) a crossed tail $T_{X'}$ with parameters $(-d_{X'}, -d_X + \frac{1}{2\lvert X\rvert}, 2\lvert X\rvert)$ for each I_K-orbit of twins $X' < X$ with $\epsilon_{X'} = -1$.

Remark 9.4. There is also a description of the action of $\mathrm{Gal}(\bar{k}/k)$ on the special fibre in terms of clusters. Moreover, one can in principle find equations for the components of the special fibre. We refer to the references below.

Example 9.5 (A type II* elliptic curve). Take $E : y^2 = x^3 - p^5$ over $\mathbb{Q}_p$ for $p \geqslant 5$, and ζ_3 a primitive third root of unity.[†] The cluster picture is



with $\quad \mathcal{R} = \{p^{\frac{5}{3}}, \zeta_3 p^{\frac{5}{3}}, \zeta_3^2 p^{\frac{5}{3}}\},$

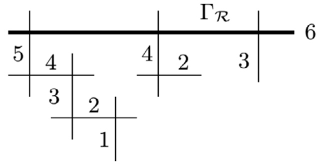
$d_\mathcal{R} = 5/3$, $\nu_\mathcal{R} = 5$, $e_\mathcal{R} = 6$ and $\lambda_\mathcal{R} = 5/2$. As $\mathbb{Q}_p(\mathcal{R})/\mathbb{Q}_p$ is tamely ramified, E has tame reduction. The cluster $\mathcal{R}$ is principal and fixed by I_K, but the roots lie in a single I_K-orbit. The special fibre of the minimal regular SNC model (displayed right) has a single central component $\Gamma_\mathcal{R}$ of multiplicity 6 and genus 0, intersected by sloped chains with parameters $(-1, 5/6, 1)$, $(-3, -5/2, 3)$, and $(-5/2, -5/3, 2)$ coming from the first, fourth, and fifth rows of the (second) table in Theorem 9.3

[†] The material in this section applies verbatim to elliptic curves of the form $y^2 =$ cubic.

respectively. By considering the sequences

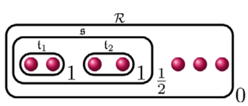
$$\frac{5}{6} > \frac{4}{5} > \frac{3}{4} > \frac{2}{3} > \frac{1}{2} > 0 > -1, \qquad -\frac{15}{2} > -8 > -9, \quad \text{and} \quad -\frac{10}{3} > -\frac{7}{2} > -4 > -5,$$

which are each minimal† satisfying the determinant condition of Definition 9.2. We find that the special fibre has the pictured form, so the Kodaira type of E is II^*.



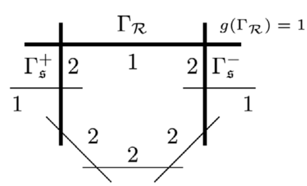
Remark 9.6. The other Kodaira types arise similarly to the above example, with one central component met by several sloped chains.

Example 9.7. Take $C/\mathbb{Q}_p : y^2 = ((x^2-p)^2 - p^4)(x^2+1)(x-1)$, cluster picture



with $\mathcal{R} = \{(p \pm p^2)^{\frac{1}{2}}, -(p \pm p^2)^{\frac{1}{2}}, i, -i, 1\}$.

The special fibre of the minimal regular SNC model (displayed below") has three central components $\Gamma_{\mathcal{R}}$ and $\Gamma_{\mathfrak{s}}^{\pm}$ ($\mathfrak{s}$ is übereven and $\epsilon_{\{\mathfrak{s}\}} = 1$). The component $\Gamma_{\mathfrak{s}}^{+}$ (respectively, $\Gamma_{\mathfrak{s}}^{-}$) intersects $\Gamma_{\mathcal{R}}$ as they are linked by a chain with parameters $(0, \frac{1}{2}, 1)$ which is empty. The $\Gamma_{\mathfrak{s}}^{\pm}$ are linked by a chain with parameters $(-1/2, 3/2, 2)$, consisting of three curves of multiplicity 2, coming from the inertia orbit $X = \{\mathfrak{t}_1, \mathfrak{t}_2\}$ with $\epsilon_X = 1$.

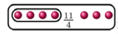


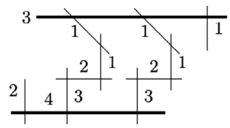
The $\Gamma_{\mathfrak{s}}^{\pm}$ are each intersected by one further chain with parameters $(-2, -1/2, 1)$ arising from the sixth row of the second table in Theorem 9.3.

References. 9.1: [14, Table 3]. 9.2: [14, Section 4.3]. 9.3: [14, Theorems 7.12 and 7.18]. 9.4: [14, Theorem 1.17, Remark 7.13]. 9.5, 9.6: [14, Example 1.13].

Erratum. In [14] Theorems 1.12 and 7.18, there is a typo : the column 't_2' in the second table of each should be '$t_2 + \delta$'. Similarly in Theorem 6.3 in the first table. For example, the curve

† See [8, Remark 3.15] for criteria guaranteeing minimality.

$C\,:\,y^2=(x^3-p^2)(x^4-p^{11})$ has cluster picture  and special fibre shown below.



10 | TAMAGAWA NUMBER (SEMISTABLE CASE)

Let $C/K\,:\,y^2=f(x)$ be a *semistable* hyperelliptic curve. The *Tamagawa number* $c_{\mathrm{Jac}\,C}$ of the Jacobian of C is the number of k-points of the component group-scheme of the special fibre of the Néron model of Jac C. We explain how to read off $c_{\mathrm{Jac}\,C}$ from the cluster picture of C.

Theorem 10.1. *Suppose that C has no übereven clusters. For even clusters $\mathfrak{s}\neq\mathcal{R}$, write $c_{\mathfrak{s}}=\begin{cases}2\delta_{\mathfrak{s}} & \text{if } \epsilon_{\mathfrak{s}}(\mathrm{Frob}^{q_{\mathfrak{s}}})=+1\\ \gcd(2\delta_{\mathfrak{s}},2) & \text{if } \epsilon_{\mathfrak{s}}(\mathrm{Frob}^{q_{\mathfrak{s}}})=-1\end{cases}$, where $q_{\mathfrak{s}}$ is the size of the* Frob-*orbit of $\mathfrak{s}$. The Tamagawa number of* Jac C *is given by*

$$c_{\mathrm{Jac}\,C}=\prod\nolimits_{\mathfrak{s}}c_{\mathfrak{s}},$$

the product taken over representatives of Frob-*orbits of even clusters $\mathfrak{s}\neq\mathcal{R}$.*

In general, when C has übereven clusters, the formula becomes significantly more complicated, and is best phrased in the language of BY trees.

Notation 10.2. Let $T=T_C$ be the BY tree associated to C (Definition 4.3), with edge-length function δ. Let B be the subgraph of T consisting of blue vertices and blue edges, and $(F,\epsilon)\in\mathrm{Aut}\,T$ (see Definition 4.5) the induced action of Frob on T.

For a vertex $v\in T\setminus B$, we write q_v for the size of the F-orbit containing v. We write $\epsilon_v=\prod_{j=0}^{q_v-1}\epsilon(F^jX)$, where X is the connected component of $T\setminus B$ containing v. If $e\in T\setminus B$ is an edge, then we define q_e and ϵ_e similarly. We write $\hat{B}\subseteq T$ for the subgraph consisting of B together with all vertices v with $\epsilon_v=-1$ and edges e with $\epsilon_e=-1$. Finally, we write $B'\subseteq\hat{B}'\subseteq T'$ for the respective quotients of $B\subseteq\hat{B}\subseteq T$ by the action of F, with length function $\delta'(e')=\delta(e)$ and with $q_{e'}=q_e$ for any edge $e\in T$ mapping to e'.

Theorem 10.3. *The Tamagawa number of* Jac C *is given by*

$$c_{\mathrm{Jac}\,C}=Q\cdot\tilde{c}\cdot\sum_{\{e'_1,\dots,e'_r\}\in R}\ \prod_{j=1}^{r}\frac{\delta'(e'_j)}{q_{e'_j}},\qquad \textit{where:}$$

(1) *Q is the product of the sizes of all F-orbits of connected components of $\hat{B}$;*
(2) *$\tilde{c}=\prod_X\tilde{c}(X)$ is a product over the connected components X of $\hat{B}'\setminus B'$ with*
  (a) *$\tilde{c}(X)=2^{\alpha-1}$ if the closure of X contains $\alpha>0$ points of B' lying an even distance from a vertex of $\hat{B}'$ of degree $\geqslant 3$;*

(b) $\tilde{c}(X) = \gcd(l, 2)$ *if the closure of* X *consists of 2 points of* B' *distance* l *apart;*

(c) $\tilde{c}(X) = \gcd(b, 2)$ *otherwise, where* b *is the number of points of* B' *in the closure of* X*;*

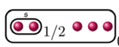
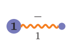
(3) $r = \#\pi_0(\hat{B}') - 1$ *is the number of connected components of* $\hat{B}'$*, minus 1;*

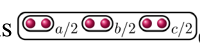
(4) R *is the set of unordered* r*-tuples of edges of* $T' \setminus \hat{B}'$ *whose removal disconnects the* $r+1$ *components of* $\hat{B}'$ *from one another.*

Remark 10.4. Theorem 10.1 follows from Theorem 10.3. Since there are no übereven clusters there are no yellow vertices, and hence all yellow edges are disjoint and in bijection with even clusters. This implies that the closures of connected components of $\hat{B}' \setminus B'$ will always consist of two vertices, distance $2\delta_{\mathfrak{s}}$ apart, and the $\tilde{c}(X)$ all fall in situation (2)(b). This is the contribution of orbits of even clusters with $\epsilon_{\mathfrak{s}}(\mathrm{Frob}^{q_{\mathfrak{s}}}) = -1$ in the formula of 10.3. Furthermore, R has size 1, the r-tuple of edges of $T' \setminus \hat{B}'$ which correspond to orbits of even clusters with $\epsilon_{\mathfrak{s}}(\mathrm{Frob}^{q_{\mathfrak{s}}}) = 1$, and so $Q \prod \delta'(e'_j)/q_{e'_j}$ is the contribution from clusters with $\epsilon_{\mathfrak{s}}(\mathrm{Frob}^{q_{\mathfrak{s}}}) = 1$.

Example 10.5. Consider

$$C : y^2 = (x^2 - 5)(x - 1)(x - 2)(x + 1),$$

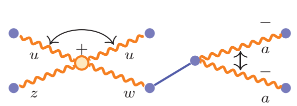
over $\mathbb{Q}_5$. Its cluster picture is  with $\mathcal{R} = \{\sqrt{5}, -\sqrt{5}, 1, 2, -1\}$ and $\mathfrak{s} = \{\sqrt{5}, -\sqrt{5}\}$. Then $\theta_{\mathfrak{s}} = \sqrt{2}$, and $\epsilon_{\mathfrak{s}}(\mathrm{Frob}) = -1$, as $\sqrt{2} \notin \mathbb{Q}_5$. According to Theorem 10.1, the Tamagawa number of Jac C is $\gcd(1, 2) = 1$. The same value can be read off from the more general Theorem 10.3, using that the BY tree of C is  with trivial F-action.

Example 10.6. Suppose that the cluster picture of C is , with Frob acting trivially on clusters and $\epsilon_{\mathfrak{s}}(\mathrm{Frob}) = +1$ for all clusters. In particular, $\hat{B} = B$ and the quotients B', T' can be identified respectively with B, T. Using Theorem 10.3, we find that the Tamagawa number of Jac C is $ab + bc + ca$ (we leave details to the reader).

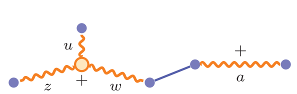
Example 10.7. Let $C/\mathbb{Q}_p$ be a curve with BY tree as below. A concrete example would be $C/\mathbb{Q}_p$ with $p \equiv 3 \bmod 4$ and

$$C : y^2 = ((x^2+1)^2 - p^u)((x-1)^2 - p^z)(x - p^{w/2})(x - p^{w/2+2})((x^2 + p^{w+4})^2 - p^{2(w+4)+a}),$$

with $w \equiv 2 \bmod 4$ and $a > w + 4$.



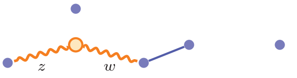
Label the edges $e_u^{\pm}, e_w, e_z, e_a^{\pm}$ where e_w has length w and so on. Since $\epsilon_v = \epsilon_e = 1$ for all vertices v and edges e, $\hat{B} = B$, and T' and B' are given by the following picture, with $\hat{B}' = B'$:



There are four F-orbits in $\hat{B}$, two of size 1 and two of size 2. Therefore, $Q = 4$. The set $\hat{B}' \setminus B'$ is empty and so $\tilde{c} = 1$. Finally, $r = 3$, and the set $R = \{\{e'_a, e'_u, e'_z\}, \{e'_a, e'_z, e'_w\}, \{e'_a, e'_w, e'_u\}\}$ where e'_a is the image of $e_a^{\pm}$ and so on. Putting this together, we see

$$c_{\mathrm{Jac}\ C} = 2\cdot 1\cdot\left(\frac{a}{2}\cdot\frac{u}{2}\cdot z+\frac{a}{2}\cdot z\cdot w+\frac{a}{2}\cdot\frac{u}{2}\cdot w\right) = a(uz+2zw+uw).$$

Example 10.8. Consider the BY tree as in Example 10.7, but where $\epsilon = -1$ for each component instead of 1. The edges $e_a^{\pm}$ and $e_u^{\pm}$ lie in F-orbits of size 2 so $\epsilon_{e_a^{\pm}} = \epsilon_{e_b^{\pm}} = 1$, and e_z and e_w lie in an orbit of size 1 so $\epsilon_{e_w} = \epsilon_{e_z} = -1$. The graphs B' and T' are as above, and $\hat{B}'$ is given by



There are three F-orbits of components in $\hat{B}$, one of size 1 and two of size 2, and hence $Q = 4$. There is one connected component $X' \in \hat{B}' \setminus B'$ and so $\tilde{c} = \tilde{c}(X')$ is non-trivial. We have assumed that w is even, and so $\tilde{c} = \gcd(z+w, 2) = \gcd(z, 2)$ as X' consists of two points of B' a distance $z+w$ apart. Finally, $r = 2$ and $R = \{\{e_a, e_u\}\}$. Putting this all together

$$c_{\mathrm{Jac}\ C} = 4\cdot\frac{a}{2}\cdot\frac{u}{2}\cdot\gcd(z,2) = au\gcd(z,2).$$

References. 10.3:[3, Theorem 3.0.2, Section 2.3.13].

11 | GALOIS REPRESENTATION

In this section, we will describe the Galois action on the ℓ-adic étale cohomology of the curve (equivalently its Jacobian) when $\ell \neq p$. For an arbitrary curve (or abelian variety), there always exists a decomposition of ℓ-adic Galois representations

$$H^1_{\text{ét}}(C/\bar{K}, \mathbb{Q}_\ell) = H^1_{\text{ét}}(\mathrm{Jac}\ C/\bar{K}, \mathbb{Q}_\ell) = H^1_{ab} \oplus \left(H^1_t \otimes \mathrm{sp}(2)\right)$$

into 'abelian' and 'toric' parts, where for $\sigma \in I_K$, $\mathrm{sp}(2)(\mathrm{Frob}^n\sigma) = \left(\begin{smallmatrix}1 & t_\ell(\sigma)\\ 0 & q^{-n}\end{smallmatrix}\right)$ for a choice of tame ℓ-adic character t_ℓ, and with $q = |k|$. We will describe the abelian and toric parts in terms of the cluster picture.

Theorem 11.1. *Let C/K be a hyperelliptic curve and let $\ell \neq p$ be prime. Let*

$$X = \{\textit{proper, non-übereven clusters } \mathfrak{s}\},$$

$$Y = \{\textit{principal, non-übereven clusters } \mathfrak{s}\} \subseteq X.$$

Write $\bar{\epsilon}_{\mathfrak{s}}$ for the restriction of $\epsilon_{\mathfrak{s}}$ to $G_{\mathfrak{s}}$.

(1) *For all $\mathfrak{s} \in Y$, there exists a continuous ℓ-adic representation, $V_{\mathfrak{s}}$, with finite image of inertia such that:*

$$H^1_{ab} = \bigoplus_{\mathfrak{s}\in Y/G_K} \mathrm{Ind}_{G_{\mathfrak{s}}}^{G_K} V_{\mathfrak{s}},$$

$$H^1_t = \bigoplus_{\mathfrak{s} \in X/G_K} \mathrm{Ind}_{G_{\mathfrak{s}}}^{G_K} \bar{\epsilon}_{\mathfrak{s}} \qquad \ominus \bar{\epsilon}_{\mathcal{R}},$$

where $G_{\mathfrak{s}} = \mathrm{Stab}_{G_K}(\mathfrak{s})$ is the Galois stabiliser of $\mathfrak{s}$, and $\ominus$ is the inverse of the direct sum $\oplus$.

(2) *Let $I_{\mathfrak{s}} = \mathrm{Stab}_{I_K}(\mathfrak{s})$ be the inertia stabiliser of $\mathfrak{s}$ and let $\gamma_{\mathfrak{s}} : I_{\mathfrak{s}} \to \bar{\mathbb{Q}}_\ell^\times$ be any character whose order is the prime-to-p part of the denominator of $[I_K : I_{\mathfrak{s}}]\tilde{\lambda}_{\mathfrak{s}}$. Then for all $\mathfrak{s} \in Y$, there is an isomorphism*

$$V_{\mathfrak{s}} \cong \gamma_{\mathfrak{s}} \otimes (\mathbb{Q}_\ell[\tilde{\mathfrak{s}}] \ominus \mathbf{1}) \quad \ominus \bar{\epsilon}_{\mathfrak{s}} \qquad \textit{as } I_{\mathfrak{s}}\textit{-representations,}$$

where $\tilde{\mathfrak{s}}$ is the set of odd children of $\mathfrak{s}$ with $I_{\mathfrak{s}}$-action.

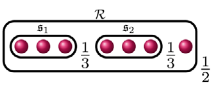
Remark 11.2. The full Galois module structure of $V_{\mathfrak{s}}$ cannot be determined by the cluster picture alone; indeed two curves with good reduction can have the same cluster picture but different Galois representations. It is, however, computable via a form of point-counting over finite fields; see [11, Theorem 1.5 and Example 1.9] for the statement and a worked example.

On the other hand, Theorem 11.1(2) gives an explicit description of the inertia representation. For tame curves, it is completely determined by the underlying abstract cluster picture (in the sense of Section 17) without needing to know the inertia action on the roots a priori.

Remark 11.3. The Jacobian Jac C (equivalently C) is semistable if and only if both H^1_{ab} and H^1_t are unramified. If moreover H^1_t is the zero representation, then this is equivalent to Jac C having good reduction. Recall from Section 5 that these conditions are easy to read off from the cluster picture of C.

Notation 11.4. For a cluster $\mathfrak{s}$, we let $I_{\mathfrak{s}}$ denote the inertia stabiliser. If n is coprime to p, we further write $\bar{\mathbb{Q}}_\ell[C_{n,\mathfrak{s}}]$ to mean the $I_{\mathfrak{s}}$-representation $\bar{\mathbb{Q}}_\ell[I_{\mathfrak{s}}/I_n]$ where $[I_{\mathfrak{s}} : I_n] = n$, and let $\chi_{n,\mathfrak{s}}$ be a fixed faithful character of $I_{\mathfrak{s}}/I_n$. We shall omit the cluster subscript when $\mathfrak{s} = \mathcal{R}$.

Example 11.5. Let ζ_3 be a primitive cube root of unity and consider the curve $C/\mathbb{Q}_7 : y^2 = x((x - 7^{1/2})^3 - 7^{5/2})((x + 7^{1/2})^3 + 7^{5/2})$, with cluster picture


$, \mathfrak{s}_1 = \{7^{1/2} + \zeta_3^j 7^{5/6} \mid j = 0, 1, 2\}, \mathfrak{s}_2 = \{-7^{1/2} - \zeta_3^j 7^{5/6} \mid j = 0, 1, 2\},$

and $\mathcal{R} = \mathfrak{s}_1 \cup \mathfrak{s}_2 \cup \{0\}$. In this case, inertia acts on $\mathcal{R}$ through a C_6-quotient and permutes $\mathfrak{s}_1$ and $\mathfrak{s}_2$. We shall compute the inertia action on $H^1_{\text{ét}}(C/\bar{\mathbb{Q}}_7, \bar{\mathbb{Q}}_\ell)$. Note that every cluster is odd: this implies that there is no contribution from the toric part, that is, $H^1_t = 0$, and also that $V_{\mathfrak{s}} \cong \gamma_{\mathfrak{s}} \otimes (\bar{\mathbb{Q}}_\ell[\tilde{\mathfrak{s}}] \ominus \mathbf{1})$ by definition of $\epsilon_{\mathfrak{s}}$. Moreover, every proper cluster is principal and hence we choose representatives for $Y/I_{\mathbb{Q}_7}$ to be $\mathfrak{s}_1$ and $\mathcal{R}$.

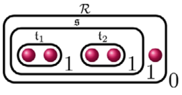
First consider $V_{\mathcal{R}}$. We compute that $\tilde{\lambda}_{\mathcal{R}} = 3/4$ hence $\gamma_{\mathcal{R}}$ is an order 4 character χ_4. Therefore $\bar{\mathbb{Q}}_\ell[\tilde{\mathcal{R}}] \cong \mathbf{1} \oplus \bar{\mathbb{Q}}_\ell[C_2]$ and hence $V_{\mathcal{R}} \cong \chi_4 \otimes \bar{\mathbb{Q}}_\ell[C_2] \cong \chi_4 \oplus \chi_4^{-1}$.

Next we compute $V_{\mathfrak{s}_1}$. In this case, $\tilde{\lambda}_{\mathfrak{s}_1} = 9/4$ and hence $\gamma_{\mathfrak{s}_1}$ is an order 2 character $\chi_{2,\mathfrak{s}_1}$ since $[I_{\mathbb{Q}_7} : I_{\mathfrak{s}_1}] = 2$. Now $I_{\mathfrak{s}_1}$ cyclically permutes the children of $\mathfrak{s}_1$ so $\bar{\mathbb{Q}}_\ell[\tilde{\mathfrak{s}}_1] \cong \bar{\mathbb{Q}}_\ell[C_{3,\mathfrak{s}_1}] \cong \mathbf{1} \oplus \chi_{3,\mathfrak{s}_1} \oplus \chi_{3,\mathfrak{s}_1}^{-1}$, hence $V_{\mathfrak{s}_1} \cong \chi_{6,\mathfrak{s}_1} \oplus \chi_{6,\mathfrak{s}_1}^{-1}$.

We must now induce this to $I_{\mathbb{Q}_7}$; since $I_{\mathbb{Q}_7}/I_{\mathfrak{s}_1} \cong C_2$ we find that $\mathrm{Ind}_{I_{\mathfrak{s}_1}}^{I_{\mathbb{Q}_7}} V_{\mathfrak{s}_1} \cong \chi_{12} \oplus \chi_{12}^5 \oplus \chi_{12}^7 \oplus \chi_{12}^{11}$. One therefore has that for all $\ell \neq 7$,

$$H^1_{\text{ét}}(C/\bar{\mathbb{Q}}_7, \bar{\mathbb{Q}}_\ell) = \chi_4 \oplus \chi_4^{-1} \oplus \chi_{12} \oplus \chi_{12}^5 \oplus \chi_{12}^7 \oplus \chi_{12}^{11}, \qquad \text{as } I_{\mathbb{Q}_7}\text{-representations.}$$

Example 11.6. Let $C/\mathbb{Q}_3$ be the curve $y^2 = (x-1)((x-3)^2+81)((x+3)^2+81)$, whose cluster picture is



where $\mathfrak{t}_1 = \{3 \pm 9i\}, \mathfrak{t}_2 = \{-3 \pm 9i\}, \mathfrak{s} = \mathfrak{t}_1 \cup \mathfrak{t}_2, \mathcal{R} = \mathfrak{s} \cup \{1\}$,

and i is a fixed square root of -1. One can check that C is semistable (see Theorem 5.1) and we shall confirm this on Galois representation side via Remark 11.3.

Note that Galois acts trivially on the proper clusters and the only übereven cluster is $\mathfrak{s}$ so $X/G_{\mathbb{Q}_3} = \{\mathfrak{t}_1, \mathfrak{t}_2, \mathcal{R}\}$. Moreover, none of these are principal ($\mathcal{R}$ is a cotwin) hence the abelian part is 0; this is expected as the Jacobian has totally toric reduction (cf. Theorem 5.6), and so $H^1_{\text{ét}}(C/\bar{\mathbb{Q}}_3, \mathbb{Q}_\ell) = (\epsilon_{\mathfrak{t}_1} \oplus \epsilon_{\mathfrak{t}_2}) \otimes \mathrm{sp}(2)$.

Using $z_{\mathfrak{t}_1} = 3$ as a centre of $\mathfrak{t}_1$, one computes that $\theta^2_{\mathfrak{t}_1} = 234 = 2 \cdot 3^2 \cdot 13$. This implies that $\theta_{\mathfrak{t}_1}$ is fixed by inertia and negated by Frobenius and therefore $\epsilon_{\mathfrak{t}_1}$ is the unramified quadratic character η. Similarly, we find that $\theta^2_{\mathfrak{t}_2} = -468$ (using the centre -3) and hence $\epsilon_{\mathfrak{t}_2} = \epsilon_{\mathfrak{t}_1} = \eta$. Since H^1_{ab} and H^1_t are both unramified, the curve is semistable (as expected) and we have that for all $\ell \neq 3$,

$$H^1_{\text{ét}}(C/\bar{\mathbb{Q}}_3, \mathbb{Q}_\ell) = \eta^{\oplus 2} \otimes \mathrm{sp}(2) \qquad \text{as } G_{\mathbb{Q}_3}\text{-representations.}$$

References. 11.1: [9, Theorem 1.19]. 11.2 ¶2, recovering the inertia action on roots: [5, Corollary 1.5].

12 | CONDUCTOR

In this section, we describe the conductor exponent of Jac C, which we shall denote by n_C.

Theorem 12.1. *Suppose C/K is semistable. Then*

$$n_C = \begin{cases} \#A - 1 & \text{if } \mathcal{R} \text{ is übereven,} \\ \#A & \text{else,} \end{cases}$$

where $A = \{\text{even clusters } \mathfrak{s} \neq \mathcal{R} \mid \mathfrak{s} \text{ is not übereven}\}$.

For general C, the formula for the conductor is more involved.

Notation 12.2. For a proper cluster $\mathfrak{s}$ we define $\xi_\mathfrak{s}(a)$ to be the 2-adic valuation of the denominator of $[I_K : I_\mathfrak{s}]a$, where $I_\mathfrak{s}$ is the stabiliser of $\mathfrak{s}$ under I_K, with the convention that $\xi_\mathfrak{s}(0) = 0$. More formally, it is $\xi_\mathfrak{s}(a) = \max\{-\mathrm{ord}_2([I_K : I_\mathfrak{s}]a), 0\}$.

For a cluster picture associated to a curve C/K, we further define

$$U = \{\text{odd clusters } \mathfrak{s} \neq \mathcal{R} \mid \xi_{P(\mathfrak{s})}(\tilde{\lambda}_{P(\mathfrak{s})}) \leqslant \xi_{P(\mathfrak{s})}(d_{P(\mathfrak{s})})\},$$

$$V = \{\text{proper non-übereven clusters } \mathfrak{s} \mid \xi_{\mathfrak{s}}(\tilde{\lambda}_{\mathfrak{s}}) = 0\}.$$

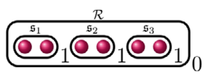
Theorem 12.3. *Let C/K be a hyperelliptic curve. Decompose the conductor exponent n_C of* Jac C *into tame and wild parts as $n_C = n_{\text{tame}} + n_{\text{wild}}$. Then:*

(1) $n_{\text{tame}} = 2g - \#(U/I_K) + \#(V/I_K) + \begin{cases} 1 & \text{if } |\mathcal{R}| \text{ and } v(c) \text{ are even,} \\ 0 & \text{else;} \end{cases}$

(2) $n_{\text{wild}} = \sum_{r \in \mathcal{R}/G_K} (v(\Delta_{K(r)/K}) - [K(r) : K] + f_{K(r)/K})$, *where $\Delta_{K(r)/K}$ and $f_{K(r)/K}$ are the discriminant and residue degree of $K(r)/K$, respectively.*

Remark 12.4. If $p > 2g+1$, then C is tame so that $n_{\text{wild}} = 0$ and $n_C = n_{\text{tame}}$. Moreover, in this case n_C is completely determined by the underlying abstract cluster picture (in the sense of Section 17) without needing to know the Galois action on the roots a priori.

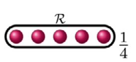
Example 12.5. Let $C/\mathbb{Q}_p : y^2 = (x^2 - p^2)((x-1)^2 - p^2)((x-2)^2 - p^2)$ with cluster picture



and $\mathcal{R} = \{p, -p, 1+p, 1-p, 2+p, 2-p\}$.

One can check that the curve is semistable (Theorem 5.1) so we can apply Theorem 12.1. Observe that $A = \{\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3\}$ from which we obtain that $n_C = 2$ since $\mathcal{R}$ is übereven.

Example 12.6. Let $C/\mathbb{Q}_5 : y^2 = x^5 + 256$ and let ζ_5 be a primitive fifth root of unity. Then the cluster picture is

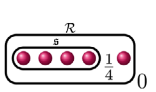


with $\mathcal{R} = \{\zeta_5^j \sqrt[5]{-256} \mid j = 0, \cdots, 4\}$.

We begin with n_{wild} and observe that the roots form a single orbit under inertia. For all $r \in \mathcal{R}$, we have that $\mathbb{Q}_5(r)/\mathbb{Q}_5$ has discriminant 50000, degree 5, residue degree 1 hence $n_{\text{wild}} = 1$.

It remains to compute n_{tame}. Now $\tilde{\lambda}_{\mathcal{R}} = \frac{5}{8}$ so $\xi_{\mathcal{R}}(\tilde{\lambda}_{\mathcal{R}}) = 3$ and $\xi_{\mathcal{R}}(d_{\mathcal{R}}) = 2$ hence U and V are both empty. Therefore $n_{\text{tame}} = 2g = 4$ and $n_C = 4 + 1 = 5$.

Example 12.7. In this example, we compute the conductor directly from the cluster picture without reference to a curve. Let $p \geqslant 7$ and let C/K be a genus two hyperelliptic curve with $c = 1$, with cluster picture



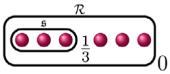
where $\mathfrak{s} = \{r_1, r_2, r_3, r_4\}$, $\mathcal{R} = \mathfrak{s} \cup \{r_5\}$

and inertia acts by cyclically permuting the roots[†] in $\mathfrak{s}$.

[†] This is actually the only possible action due to the depths. An example of such a curve is $C/\mathbb{Q}_7 : y^2 = x^5 + 5x^4 + 40x^3 + 80x^2 + 256x$.

Now we compute that $\tilde{\lambda}_{\mathcal{R}} = 0$ and $\tilde{\lambda}_{\mathfrak{s}} = \frac{1}{2}$ hence $\xi_{\mathcal{R}}(\tilde{\lambda}_{\mathcal{R}}) = \xi_{\mathcal{R}}(d_{\mathcal{R}}) = 0$, $\xi_{\mathcal{R}}(\tilde{\lambda}_{\mathfrak{s}}) = 1$ and $\xi_{\mathcal{R}}(d_{\mathfrak{s}}) = 2$. This means that $U/I_{\mathbb{Q}_5} = \{\{r_1\},\{r_5\}\}$ (since r_2, r_3, r_4 are conjugate to r_1) and $V/I_{\mathbb{Q}_5} = \{\mathcal{R}\}$. Thus $n_C = 4 - 2 + 1 = 3$ by Remark 12.4.

Example 12.8. Let $C/\mathbb{Q}_{97} : y^2 = (x^3 - 97)(x-1)(x-2)(x-3)$ with cluster picture



where $\mathfrak{s} = \{\zeta_3^j \sqrt[3]{97} \mid j = 0,1,2\}$, and $\zeta_3^3 = 1 \neq \zeta_3$.

Again by Remark 12.4, $n_C = n_{\text{tame}}$. Now $\tilde{\lambda}_{\mathcal{R}} = 0$ and $\tilde{\lambda}_{\mathfrak{s}} = \frac{1}{2}$ from which we see that $U/I_{\mathbb{Q}_{97}} = \{\mathfrak{s}, \{1\}, \{2\}, \{3\}\}$ and $V/I_{\mathbb{Q}_{97}} = \{\mathcal{R}\}$. Therefore $n_C = 4 - 4 + 1 + 1 = 2$ since $|\mathcal{R}|$ and $v(c)$ are both even.

Remark 12.9. As the first and last examples show, the conductor does not determine whether a curve is semistable, in contrast to the elliptic curve setting.

References. 12.1, 12.3 : [9, Theorem 1.20]. 12.4: Remark 5.7, [5, Corollary 1.5].

13 | ROOT NUMBER (TAME CASE)

We give a description of the local root number $W(A/K)$ of an abelian variety A (for example, $A = \text{Jac}\, C$), first in the case of semistable reduction, then in the case of tame reduction. For Jac C, we give this description in terms of the cluster picture.

Notation 13.1. Throughout, χ_n will denote a fixed character of I_K of order n, and for an abelian variety A/K we shall decompose $H^1_{\text{ét}}(A/\bar{K}, \mathbb{Q}_\ell) = H^1_{ab} \oplus (H^1_t \otimes \text{sp}(2))$ as in Section 11. For a cluster $\mathfrak{s}$, let $G_{\mathfrak{s}} = \text{Stab}_{G_K}(\mathfrak{s})$ and $I_{\mathfrak{s}} = \text{Stab}_{I_K}(\mathfrak{s})$ be its Galois and inertia stabilisers, respectively, and let $n_{\mathfrak{s}} = [I_K : I_{\mathfrak{s}}]$. We write X for the set of all cotwins and all even, non-übereven clusters of C.

Theorem 13.2. *Let A/K be an abelian variety with semistable reduction. Then*

$$W(A/K) = (-1)^{\langle \mathbf{1}, H^1_t \rangle}.$$

When C/K is semistable, $W(\text{Jac}\, C/K)$ may then be computed from the cluster picture as follows.

Proposition 13.3. *Let C/K be a (not necessarily semistable) hyperelliptic curve. The toric part ρ_t of the representation of* Jac C *satisfies*

$$\langle \mathbf{1}, \rho_t \rangle = \#\{\mathfrak{s} \in (X \setminus \mathcal{R})/G_K : \text{Res}_{G_{\mathfrak{s}}} \epsilon_{\mathfrak{s}} = \mathbf{1}\} - \begin{cases} 1 & \mathcal{R} \text{ übereven and } c \in K^{\times 2}, \\ 0 & \text{else.} \end{cases}$$

Theorem 13.4. *Let A/K be an abelian variety with tame reduction. Let*

$$m_t = \langle H^1_t|_{I_K}, \chi_2 \rangle, \qquad m_e = \langle H^1_{ab}|_{I_K}, \chi_e \rangle \text{for } e \geq 2.$$

Then

$$W(A/K) = \left(\prod_{e\geq 3} W_{q,e}^{m_e}\right)(-1)^{\langle \mathbf{1}, H_t^1\rangle} W_{q,2}^{m_t+\frac{1}{2}m_2},$$

where $q = |k|$ *and*

$$W_{q,e} = \begin{cases} \left(\frac{q}{l}\right) & \text{if } e = l^n \text{ for some odd prime } l \text{ and integer } n \geq 1;\\ \left(\frac{-1}{q}\right) & \text{if } e = 2l^n \text{ for some prime } l \equiv 3 \bmod 4,\ n \geq 1,\ \text{or if } e = 2;\\ \left(\frac{-2}{q}\right) & \text{if } e = 4;\\ \left(\frac{2}{q}\right) & \text{if } e = 2^n \text{ for some integer } n \geq 3;\\ 1 & \text{else.}\end{cases}$$

In the case of Jacobians of hyperelliptic curves with tame reduction, $\langle \mathbf{1}, H_t^1\rangle$ can be calculated as in Proposition 13.3 and m_t can be read off the cluster picture.

Proposition 13.5. *Let C/K be a hyperelliptic curve with tame reduction and* $\deg(f)$ *even. Then*

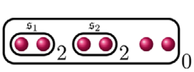
$$m_t \equiv v(c) + \#\{\mathfrak{s} \in X/I_K \ :\ n_{\mathfrak{s}}(\nu_{\mathfrak{s}} - |\mathfrak{s}|d_{\mathfrak{s}}) \text{ even}\} + \sum_{\mathfrak{s}\in X/I_K} n_{\mathfrak{s}} \mod 2.$$

The final quantities which are required are the m_e, the multiplicities of the eigenvalues of the abelian part of the representation. These are straightforward to calculate by hand in terms of the Galois representation from Section 11. An explicit but rather messy description in terms of the cluster pictures exists; see [5, Theorem 4.5, Section 4.1.2].

Remark 13.6. The root number of curves C with potentially totally toric reduction (possibly wild) can be calculated via [4, Lemma 3.8, 4.1].

Example 13.7. First we give three cases where the curve is semistable (this can be checked using Theorem 5.1), and the calculation simplifies by Theorem 13.2.

(i) Let $C_1/\mathbb{Q}_{23}$ be given by[†]



$$y^2 = (x^2 - 23^4)((x-1)^2 - 23^4)(x-2)(x-3),$$

with $\mathcal{R} = \{23^2, -23^2, 1+23^2, 1-23^2, 2, 3\}$. Note that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ lie in their own orbits, and $\epsilon_{\mathfrak{s}_1} = \epsilon_{\mathfrak{s}_2} = \mathbf{1}$. Therefore by Proposition 13.3, $\langle \mathbf{1}, H_t^1\rangle = 2$ and

$$W(\text{Jac } C_1/\mathbb{Q}_{23}) = (-1)^{\langle \mathbf{1}, H_t^1\rangle} = (-1)^2 = 1.$$

(ii) Now let $C_2/\mathbb{Q}_{23}$ be given by

$$y^2 = ((x-i)^2 - 23^4)((x+i)^2 - 23^4)(x-2)(x-3),$$

[†] We picked $p = 23$ as it is the smallest prime p such that $(\frac{2}{p}) = (\frac{3}{p}) = 1$ and $(\frac{-1}{p}) = -1$.

with $\mathcal{R} = \{i + 23^2, i - 23^2, -i + 23^2, -i - 23^2, 2, 3\}$. The cluster picture of C_2 is the same as C_1, but now Frobenius swaps $\mathfrak{s}_1$ and $\mathfrak{s}_2$. One checks that $\mathrm{Res}_{G_\mathfrak{s}} \epsilon_{\mathfrak{s}_1} = \mathbf{1}$. Therefore, $\langle \mathbf{1}, H^1_t \rangle = 1$ and hence

$$W(\mathrm{Jac}\, C_2)/\mathbb{Q}_{23}) = (-1)^{\langle \mathbf{1}, H^1_t \rangle} = (-1)^1 = -1.$$

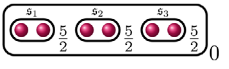
(iii) Now let $C_3/\mathbb{Q}_{23}$ be given by

$$y^2 = -(x^2 - 23^4)((x-1)^2 - 23^4)(x-2)(x-3),$$

so the roots and cluster picture are the same as (i), but now the leading coefficient is -1. Now $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are in their own orbits again but $\epsilon_{\mathfrak{s}_1}$ and $\epsilon_{\mathfrak{s}_2}$ have order 2. Therefore $\langle \mathbf{1}, H^1_t \rangle = 0$ and hence

$$W(\mathrm{Jac}\, C_3/\mathbb{Q}_{23}) = (-1)^{\langle \mathbf{1}, H^1_t \rangle} = (-1)^0 = 1.$$

Example 13.8. Let $C/\mathbb{Q}_7$ be given by

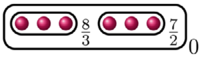


$$y^2 = 7\left(x^2 - 7^5\right)\left((x-1)^2 - 7^5\right)\left((x-2)^2 - 7^5\right).$$

Since there are no principal, non-übereven clusters, the abelian part of the representation is trivial. We calculate $\mathfrak{s}_1^* = \mathcal{R}$, $\theta^2_{\mathfrak{s}_1^*} = 7$ and hence $\epsilon_{\mathfrak{s}_1}$ is a character of order 2. Similarly, $\epsilon_{\mathfrak{s}_2}$ and $\epsilon_{\mathfrak{s}_3}$ are characters of order 2. Therefore, since $c \notin \mathbb{Q}_7^{\times 2}$, $\langle \mathbf{1}, H^1_t \rangle = 0$ by Proposition 13.3. Furthermore, $\mu = 1$ and for $i = 1, 2, 3$, $n_{\mathfrak{s}_i} = 1$ and $n_{\mathfrak{s}_i}(\nu_{\mathfrak{s}_i} - |\mathfrak{s}_i| d_{\mathfrak{s}_i}) = 1$. By Proposition 13.5, $m_t \equiv 4 \bmod 2$, so by Theorem 13.4

$$W(\mathrm{Jac}\, C/\mathbb{Q}_7) = W^4_{7,2} = 1.$$

Example 13.9. Let $C/\mathbb{Q}_7$ be given by



$$y^2 = (x^3 - 7^8)(x-1)((x-1)^2 - 7^7).$$

Since the only even cluster is $\mathcal{R}$, the toric part of the representation is trivial (Theorem 5.6) and hence only the abelian part contributes to the root number. On inertia, $H^1_{ab} = \chi_3 \oplus \chi_3^{-1} \oplus \chi_4 \oplus \chi_4^{-1}$ and so $m_3 = 1$, $m_4 = 1$ and $m_e = 0$ for all other $e \in \mathbb{N}$. We calculate $W_{7,3} = (\frac{7}{3}) = 1$, $W_{7,4} = (\frac{-2}{7}) = -1$ and

$$W(\mathrm{Jac}\, C/\mathbb{Q}_7) = W_{7,3} W_{7,4} = -1.$$

References. 13.2, 13.4: [4, Theorem 1.5]. 13.3: Theorem 11.1. 13.5: [5, Corollary 4.9, Remark 4.10].

14 | DIFFERENTIALS (SEMISTABLE CASE)

Let $\Omega^1_{C/K}(C)$ be the g-dimensional K-vector space of regular differentials of C. It is spanned by $\omega_0, \ldots, \omega_{g-1}$, where $\omega_i = x^i \frac{dx}{y}$.

Fix a regular model $\mathcal{C}/\mathcal{O}_K$ of C (see Section 7), and consider the global sections of the relative dualising sheaf $\omega_{\mathcal{C}/\mathcal{O}_K}$.

Remark 14.1.

(i) $\omega_{\mathcal{C}/\mathcal{O}_K}(\mathcal{C})$ can be thought of as the space of those differentials that are regular not only along C (the generic fibre of $\mathcal{C}$) but also along every irreducible component of the special fibre of $\mathcal{C}$.
(ii) $\omega_{\mathcal{C}/\mathcal{O}_K}(\mathcal{C})$ can be viewed as an $\mathcal{O}_K$-lattice in $\Omega^1_{C/K}(C)$.
(iii) $\omega_{\mathcal{C}/\mathcal{O}_K}(\mathcal{C})$ is independent of the choice of the regular model $\mathcal{C}$.

Definition 14.2. A *basis of integral differentials of* C, denoted $\omega_0^\circ, \dots, \omega_{g-1}^\circ$, is an $\mathcal{O}_K$-basis of $\omega_{\mathcal{C}/\mathcal{O}_K}(\mathcal{C})$ as an $\mathcal{O}_K$-lattice in $\Omega^1_{C/K}(C)$.

Theorem 14.3. *Suppose C/K is semistable. For $i = 0, \dots, g-1$ inductively*

- *compute $e_{\mathfrak{t},i} = \frac{\nu_{\mathfrak{t}}}{2} - d_{\mathfrak{t}} - \sum_{j=0}^{i-1} d_{\mathfrak{s}_j \wedge \mathfrak{t}}$ for every proper cluster $\mathfrak{t}$;*
- *choose a proper cluster $\mathfrak{s}_i$ so that $e_{\mathfrak{s}_i,i} = \max_{\mathfrak{t}}\{e_{\mathfrak{t},i}\}$.*[†] *Denote $e_{\mathfrak{s}_i,i}$ by e_i.*

Fix a centre $z_{\mathfrak{s}} \in K^{nr}$ for every proper cluster $\mathfrak{s}$,[‡] *then choose a finite unramified extension F/K such that $z_{\mathfrak{s}} \in F$ for all $\mathfrak{s}$. Let $\beta \in \mathcal{O}_F^\times$ be any element such that $\mathrm{Tr}_{F/K}(\beta) \in \mathcal{O}_K^\times$. Then the differentials*

$$\omega_i^\circ = \pi^{e_i} \cdot \mathrm{Tr}_{F/K}\left(\beta \prod_{j=0}^{i-1}(x - z_{\mathfrak{s}_j})\right)\frac{dx}{y}, \qquad i = 0, \dots, g-1,$$

form a basis of integral differentials of C.

Remark 14.4. If $F = K$, then in Theorem 14.3 we can choose $\beta = 1$ and the trace is just the identity. One can take $F = K$ if and only if Frob does not permute clusters.

Consider $\omega = \omega_0 \wedge \dots \wedge \omega_{g-1}, \omega^\circ = \omega_0^\circ \wedge \dots \wedge \omega_{g-1}^\circ \in \det \Omega^1_{C/K}(C) = \bigwedge^g \Omega^1_{C/K}(C)$. As $\det \Omega^1_{C/K}(C)$ is a 1-dimensional K-vector space, there exists $\lambda \in K$ such that $\omega^\circ = \lambda \cdot \omega$. We will denote this element by $\frac{\omega^\circ}{\omega}$.

Remark 14.5. Note that $\frac{\omega^\circ}{\omega}$ is only well defined up to a unit. Moreover, it depends on the choice of Weierstrass equation for C.

Theorem 14.6. *Suppose C/K is semistable. With the notation above,*

$$8 \cdot v\left(\frac{\omega^\circ}{\omega}\right) = 4g \cdot v(c) + \sum_{\mathfrak{s}\ \text{even}} \delta_{\mathfrak{s}}(|\mathfrak{s}| - 2)|\mathfrak{s}| + \sum_{\mathfrak{s}\ \text{odd}} \delta_{\mathfrak{s}}(|\mathfrak{s}| - 1)^2, \qquad \text{where } \delta_{\mathcal{R}} = d_{\mathcal{R}}.$$

[†] Suppose the maximal value is obtained by two different clusters $\mathfrak{s}$ and $\mathfrak{s}'$. If $\mathfrak{s}' \subseteq \mathfrak{s}$, choose $\mathfrak{s}_i = \mathfrak{s}$, if $\mathfrak{s} \subseteq \mathfrak{s}'$, choose $\mathfrak{s}_i = \mathfrak{s}'$, otherwise choose freely any of the two.

[‡] This is always possible by Theorem 5.1 and [9, Lemma B.1] since C is semistable.

Proposition 14.7. *Let Δ_C be the discriminant of C (see Section 15). Then*

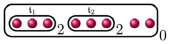
$$g \cdot v(\Delta_C) - (8g+4) \cdot v\left(\frac{\omega^\circ}{\omega}\right)$$

is independent of the choice of equation for C. If C/K is semistable, it is given by

$$\sum_{\substack{\mathfrak{s} \text{ even} \\ 1<|\mathfrak{s}|<2g+1}} \frac{\delta_{\mathfrak{s}}}{2}|\mathfrak{s}|(2g+2-|\mathfrak{s}|) + \sum_{\substack{\mathfrak{s} \text{ odd} \\ 1<|\mathfrak{s}|<2g+1}} \frac{\delta_{\mathfrak{s}}}{2}(|\mathfrak{s}|-1)(2g+1-|\mathfrak{s}|).$$

Example 14.8. Consider the semistable genus 3 curve

$$C\,:\,y^2 = ((x-7^2)^2+1)((x-2\cdot 7^2)^2+1)((x-3\cdot 7^2)^2+1)(x^2-1) \text{ over } \mathbb{Q}_7.$$

Its cluster picture is  with $\mathfrak{t}_1 = \{i+7^2,\ i+2\cdot 7^2,\ i+3\cdot 7^2\}$, $\mathfrak{t}_2 = \{-i+7^2,\ -i+2\cdot 7^2,\ -i+3\cdot 7^2\}$ and $\mathcal{R} = \mathfrak{t}_1 \cup \mathfrak{t}_2 \cup \{\pm 1\}$, where $i^2 = -1$. We want to find a basis of integral differentials of C using Theorem 14.3. First compute $e_{\mathfrak{t},0}$ for $\mathfrak{t}_1, \mathfrak{t}_2, \mathcal{R}$ and note that $e_{\mathfrak{t}_1,0} = e_{\mathfrak{t}_2,0} = \max_{\mathfrak{t}}\{e_{\mathfrak{t},0}\}$ (see table below). Since neither $\mathfrak{t}_1 \subset \mathfrak{t}_2$ nor $\mathfrak{t}_2 \subset \mathfrak{t}_1$, we are free to choose any of the two as $\mathfrak{s}_0$. Set $\mathfrak{s}_0 = \mathfrak{t}_2$. We repeat this procedure for $e_{\mathfrak{t},1}$ and $e_{\mathfrak{t},2}$ as shown in the following table.

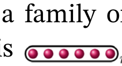
	$z_{\mathfrak{t}}$	$d_{\mathfrak{t}}$	$\nu_{\mathfrak{t}}/2$	$e_{\mathfrak{t},0}\ (=\nu_{\mathfrak{t}}/2 - d_{\mathfrak{t}})$	$e_{\mathfrak{t},1}\ (=e_{\mathfrak{t},0} - d_{\mathfrak{s}_0\wedge\mathfrak{t}})$	$e_{\mathfrak{t},2}\ (=e_{\mathfrak{t},1} - d_{\mathfrak{s}_1\wedge\mathfrak{t}})$
$\mathfrak{t}_1$	i	2	3	1	1	−1
$\mathfrak{t}_2$	$-i$	2	3	1	−1	−1
$\mathcal{R}$	0	0	0	0	0	0

The numbers coloured in red are the quantities e_i. Choosing $F = \mathbb{Q}_7(i)$, $\beta = 1$, we have

$$\begin{aligned}
\omega_0^\circ &= 7^1 \mathrm{Tr}_{\mathbb{Q}_7(i)/\mathbb{Q}_7}(1)\frac{dx}{y} = 14\frac{dx}{y}, \\
\omega_1^\circ &= 7^1 \mathrm{Tr}_{\mathbb{Q}_7(i)/\mathbb{Q}_7}(x+i)\frac{dx}{y} = 14x\frac{dx}{y}, \\
\omega_2^\circ &= 7^0 \mathrm{Tr}_{\mathbb{Q}_7(i)/\mathbb{Q}_7}((x+i)(x-i))\frac{dx}{y} = 2(x^2+1)\frac{dx}{y},
\end{aligned}
\qquad
\begin{pmatrix} \omega_0^\circ \\ \omega_1^\circ \\ \omega_2^\circ \end{pmatrix} = \begin{pmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{pmatrix},$$

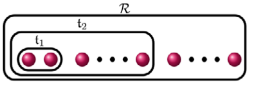
form a basis of integral differentials. In particular, $\omega^\circ = 8 \cdot 7^2 \omega$ and so $v(\frac{\omega^\circ}{\omega}) = 2$. Finally, we check this result agrees with what the formula in Theorem 14.6 predicts

$$\begin{aligned}
v\left(\frac{\omega^\circ}{\omega}\right) &= \tfrac{1}{8}\left(4g \cdot v(c) + d_{\mathcal{R}}(|\mathcal{R}|-2)|\mathcal{R}| + \delta_{\mathfrak{t}_1}(|\mathfrak{t}_1|-1)^2 + \delta_{\mathfrak{t}_2}(|\mathfrak{t}_2|-1)^2\right) \\
&= \tfrac{1}{8}\left(12 \cdot 0 + 0(8-2)8 + 2(3-1)^2 + 2(3-1)^2\right) = 2.
\end{aligned}$$

Example 14.9. Let $f(x) = 7^2(x^6-1) \in \mathbb{Q}_7[x]$ and $C^n\,:\,y^2 = 7^{6n} f(x/7^n)$, $n \in \mathbb{Z}$, a family of isomorphic semistable hyperelliptic curves of genus 2. The cluster picture of C^n is  with $\mathcal{R} = \{7^n, \zeta_3 7^n, \zeta_3^2 7^n, -7^n, -\zeta_3 7^n, -\zeta_3^2 7^n\}$. Since we have only one cluster, $\mathfrak{s}_0 = \mathfrak{s}_1 = \mathcal{R}$. Then $e_0 = 1 + 2n$ and $e_1 = 1 + n$. As 0 is a centre of $\mathcal{R}$, we are in the situation of Remark 14.4, and so $\omega_0^\circ = 7^{1+2n}\frac{dx}{y}$, $\omega_1^\circ = 7^{1+n}x\frac{dx}{y}$. This shows that $v(\omega^\circ/\omega) = 2 + 3n$ does depend on n, that is, on the choice of equation.

On the other hand, from the formula in Proposition 14.7 we immediately see that $g \cdot v(\Delta_C) - (8g+4) \cdot v(\omega^\circ/\omega) = 0$, which is independent of n.

Example 14.10. Let $C : y^2 = f(x)$ be a semistable curve with $f(x)$ monic. Suppose



is its cluster picture, with $d_{\mathfrak{t}_1} = u/2, d_{\mathfrak{t}_2} = a, d_{\mathcal{R}} = b$, for some $u, a, b \in \mathbb{Z}, u/2 > a > b$.

As in Example 14.8, to compute e_i for $i = 0, \dots g-1$, we draw the following table

	$d_{\mathfrak{t}}$	$e_{\mathfrak{t},0}$	$e_{\mathfrak{t},1}$	…	$e_{\mathfrak{t},m-1}$	$e_{\mathfrak{t},m}$	…
$\mathfrak{t}_1$	$u/2$	$\frac{\lvert\mathfrak{t}_2\rvert-\lvert\mathfrak{t}_1\rvert}{2}a + \frac{\lvert\mathcal{R}\rvert-\lvert\mathfrak{t}_2\rvert}{2}b$	$e_{\mathfrak{t}_1,0} - a$	…	$e_{\mathfrak{t}_1,0} - (m-1)a$	$e_{\mathfrak{t}_1,0} - ma$	…
$\mathfrak{t}_2$	a	$\frac{\lvert\mathfrak{t}_2\rvert-\lvert\mathfrak{t}_1\rvert}{2}a + \frac{\lvert\mathcal{R}\rvert-\lvert\mathfrak{t}_2\rvert}{2}b$	$e_{\mathfrak{t}_2,0} - a$	…	$e_{\mathfrak{t}_2,0} - (m-1)a$	$e_{\mathfrak{t}_2,0} - ma$	…
$\mathcal{R}$	b	$\frac{\lvert\mathcal{R}\rvert-\lvert\mathfrak{t}_1\rvert}{2}b$	$e_{\mathcal{R},0} - b$	…	$e_{\mathcal{R},0} - (m-1)b$	$e_{\mathcal{R},0} - mb$	…

where m is the least positive integer such that $e_{\mathcal{R},0} - mb \geqslant e_{\mathfrak{t}_2,0} - ma$. Then $m = \lfloor \frac{|\mathfrak{t}_2|-1}{2} \rfloor$ and $\mathfrak{s}_0 = \cdots = \mathfrak{s}_{m-1} = \mathfrak{t}_2, \mathfrak{s}_m = \cdots = \mathfrak{s}_{g-1} = \mathcal{R}$. Note that the twin $\mathfrak{t}_1$ is never selected, and $\omega_0^\circ, \dots, \omega_{m-1}^\circ$ form a basis of integral differentials of $C_{\mathfrak{t}_2} : y^2 = \prod_{r \in \mathfrak{t}_2}(x-r)$. These are general phenomena (see [15, Lemma 4.2]).

References. 14.1: [18, Corollaries 8.3.6(d), 5.2.27, 9.2.25]. 14.3: [20, Theorem 6.4], [15, Theorem 4.1], Theorem 5.1, [9, Lemma B.1]. 14.5, 14.6: [15, Theorem 3.1]. 14.7: [15, Proposition 3.8], Theorem 14.6, Theorem 15.1.

15 | MINIMAL DISCRIMINANT (SEMISTABLE CASE)

The *discriminant* Δ_C of C is given by

$$\Delta_C = 16^g \, c^{4g+2} \, \mathrm{disc}\Big(\frac{1}{c} f(x)\Big).$$

The following theorem provides a formula to compute the valuation of the discriminant in terms of cluster pictures.

Theorem 15.1. *The valuation of the discriminant of C is given by*

$$v(\Delta_C) = v(c)(4g+2) + \sum_{\mathfrak{s} \text{ proper}} \delta_{\mathfrak{s}} |\mathfrak{s}|(|\mathfrak{s}|-1),$$

where $\delta_{\mathfrak{s}} = d_{\mathcal{R}}$ when $\mathfrak{s} = \mathcal{R}$.

Let $v(\Delta_C^{\min})$ denote the valuation of the minimal discriminant† of the curve C. If C has semistable reduction, one may read off this quantity from the cluster picture or from the centred BY tree associated to the equation.

† The valuation of the minimal discriminant is the minimum of $v(\Delta)$ amongst all integral Weierstrass equations for C.

Theorem 15.2. *If C/K is semistable and $|k| > 2g+1$, then*

$$\frac{v(\Delta_C) - v(\Delta_C^{\min})}{4g+2} = v(c) - E + \sum_{g+1<|\mathfrak{s}|} \delta_{\mathfrak{s}}(|\mathfrak{s}| - g - 1),$$

where $\delta_{\mathfrak{s}} = d_{\mathcal{R}}$ when $\mathfrak{s} = \mathcal{R}$, and $E = 0$ unless there are two clusters of size $g+1$ that are permuted by Frobenius and $v(c)$ is odd, in which case $E = 1$.

Definition 15.3. For a connected subgraph T of a BY tree, we define a genus function by $g(T) =$ #(connected components of the blue part) $- 1 + \sum_{v \in V(T)} g(v)$.

If there is an edge $e \in E(T_C)$ such that both trees in $T_C \setminus \{e\}$ have equal genus (that is, genus $\lfloor \frac{g}{2} \rfloor$), then we insert a vertex z_T on the midpoint of e and call it the *centre* of T_C. Otherwise, there exists a unique vertex $v \in V(T_C)$ such that all trees in $T_C \setminus \{v\}$ have genus smaller than $g/2$. In this case, $z_T = v$ is the centre of T_C. In both cases, the *centred BY tree* T_C^* is the tree with vertex set $V(T_C) \cup \{z_T\}$; we denote by $\preceq$ the partial ordering on $V(T_C^*)$ with maximal element z_T.

Notation 15.4. Define a weight function on $V(T_C^*)$ by

$$S(v) = \sum_{v' \preceq v \text{ blue}} (2g(v') + 2 - \#\text{blue edges at } v').$$

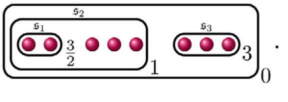
For each $v \neq z_T$, write e_v for the edge connecting v with its parent, that is, the vertex connected to v lying on the path to the centre of T_C^*. Let $\delta_v = \text{length}(e_v)$ if e_v is blue, and $\delta_v = 1/2 \cdot \text{length}(e_v)$ if e_v is yellow.

Theorem 15.5. *Suppose that C is semistable and $|k| > 2g+1$. Let T_C^* be the centred BY tree associated to C. Then the valuation of the minimal discriminant of C is given by*

$$v(\Delta_C^{\min}) = E \cdot (4g+2) + \sum_{v \neq z_T} \delta_v S(v)(S(v) - 1),$$

where $E = 0$ unless z_T has exactly two children v_1, v_2 with $S(v_1) = S(v_2) = g+1$ that are permuted by Frobenius and $(g+1)\delta_{v_1}, (g+1)\delta_{v_2}$ are odd, in which case $E = 1$.

Example 15.6. Consider $C : y^2 = p(x^2 - p^5)(x^3 - p^3)((x-1)^3 - p^9)$ over $\mathbb{Q}_p$ for $p > 7$. This is a genus 3 hyperelliptic curve with cluster picture



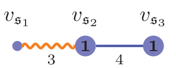
.

Using the formula from Theorem 15.1, we get that the valuation of the discriminant of the equation is

$$v(\Delta_C) = 1 \cdot (4 \cdot 3 + 2) + 3/2 \cdot 2 \cdot 1 + 1 \cdot 5 \cdot 4 + 3 \cdot 3 \cdot 2 = 55.$$

Since C has semistable reduction and $|\mathbb{F}_p| > 7$, we may now apply Theorem 15.2 in order to find the valuation of the minimal discriminant. The right-hand side of the equation in that theorem is $v(c) - E + \sum_{g+1<|\mathfrak{s}|} \delta_{\mathfrak{s}}(|\mathfrak{s}| - g - 1) = 2$, hence $v(\Delta_C^{\min}) = v(\Delta_C) - 2 \cdot (4g + 2) = 27$.

Alternatively, we could have used the associated BY tree T_C:



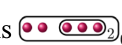
In this example, $V(T_C^*) = V(T_C)$ and $v_{\mathfrak{s}_2}$ is the centre of T_C^*. Then $S(v_{\mathfrak{s}_1}) = 2$, $S(v_{\mathfrak{s}_3}) = 3$, $\delta_{v_{\mathfrak{s}_1}} = 3/2$ and $\delta_{v_{\mathfrak{s}_3}} = 4$. It follows from Theorem 15.5 that $v(\Delta_C^{\min}) = 3/2 \cdot 2 \cdot 1 + 4 \cdot 3 \cdot 2 = 27$.

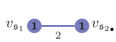
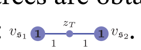
Example 15.7. Consider the curve $C : y^2 = 7(x^2 + 1)(x^2 + 36)(x^2 + 64)$ defined over $\mathbb{Q}_7$. This is a genus 2 hyperelliptic curve with cluster picture  Using one of the formulas from Theorem 15.1, we get $v(\Delta_C) = 22$.

Since C has semistable reduction, we can apply Theorem 15.2. Note that the two clusters $\mathfrak{s}_1 = \{i, i \pm 7i\}$, $\mathfrak{s}_2 = \{-i, -i \pm 7i\}$ are permuted by Frobenius. Therefore $E = 1$ here and the right-hand side of the formula vanishes. In particular, we find that $v(\Delta_C^{\min}) = v(\Delta_C) = 22$. The minimality of the equation is also implied by Theorem 16.3, since Condition (1) of that theorem is satisfied.

The minimal discriminant is not invariant under unramified extensions. Let C_K denote the base change of C to $K = \mathbb{Q}_7(i)$. Since the extension is unramified, the cluster picture does not change. However, the two clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are no longer swapped by Frobenius, hence $E = 0$ and by Theorem 15.1, $v(\Delta_{C_K}^{\min}) = v(\Delta_{C_K}) - (4g + 2) = 12$. A minimal Weierstrass equation over K can be attained by the change of variables $x = i(x' + 6)/(x' - 1)$ and $y = 49y'/(x' - 1)^3$:

$$y'^2 = -x'(x' - 2)(2x' + 5)(5x' - 12)(9x' - 2).$$

The cluster picture corresponding to this equation is .

In both of the above cases, the associated BY trees consist of two blue vertices joined by a blue edge of length 2: . The centred BY trees are obtained by adding an additional vertex in the midpoint of the edge joining $v_{\mathfrak{s}_1}$ and $v_{\mathfrak{s}_2}$: . From the formula in Theorem 15.5, we see that the valuation of the minimal discriminant is given by $12 + 10 \cdot E$. The only difference between the (centred) BY trees corresponding to C and C_K is the action of Frobenius, and we have $E = 1$ for C and $E = 0$ for C_K. As before, we find $v(\Delta_C^{\min}) = 22$ and $v(\Delta_{C_K}^{\min}) = 12$.

References. 15.1: [9, Theorem 16.2, Lemma 16.5], 15.2: [9, Theorem 16.2]. 15.3: Definitions A.1, A.2, Remark A.4. 15.5: Theorem A.6.

16 | MINIMAL WEIERSTRASS EQUATION

Here we explain how one can tell if a Weierstrass equation is minimal. Recall that a Weierstrass equation of a curve $C/K : y^2 = f(x)$ is *integral* if $f(x) \in \mathcal{O}_K[x]$. It is *minimal* if the valuation of its discriminant is minimal amongst all integral Weierstrass equations.

We first characterise when the equation is integral in terms of the cluster picture. Note that the cluster picture of hyperelliptic curve is unchanged by a substitution $x \mapsto x - t$. As a result, for a hyperelliptic curve $C/K : y^2 = f(x)$ it is not possible to check whether $f(x) \in \mathcal{O}_K[x]$ from the cluster picture of C, but up to these shifts in the x-coordinate this is possible.

Theorem 16.1. *Let $C/K : y^2 = f(x)$ be a hyperelliptic curve and suppose that G_K acts tamely on $\mathcal{R}$. Then $f(x-z) \in \mathcal{O}_K[x]$ for some $z \in K$ if and only if either*

- *$v(c) \geqslant 0$ and $d_{\mathcal{R}} \geqslant 0$; or*
- *there is a G_K-stable proper cluster $\mathfrak{s}$ with $d_{\mathfrak{s}} \leqslant 0$ and*

$$v(c) + (|\mathfrak{s}| - |\mathfrak{s}'|)d_{\mathfrak{s}} + \sum_{r \notin \mathfrak{s}} d_{\{r\} \wedge \mathfrak{s}} \geqslant 0,$$

for some $\mathfrak{s}'$ that is either empty or a G_K-stable child $\mathfrak{s}' < \mathfrak{s}$ with either $|\mathfrak{s}'| = 1$ or $d_{\mathfrak{s}'} \geqslant 0$.

We are further able to give a criterion for checking whether a given Weierstrass equation is in fact minimal.

Theorem 16.2. *Let $C : y^2 = f(x)$ be a hyperelliptic curve over K with $f(x) \in \mathcal{O}_K[x]$. If $d_{\mathcal{R}} = v(c) = 0$ and the cluster picture of C has no cluster $\mathfrak{s} \neq \mathcal{R}$ with $|\mathfrak{s}| > g+1$, then C is a minimal Weierstrass equation.*

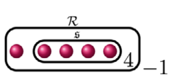
For semistable hyperelliptic curves, we can give a full characterisation of minimal Weierstrass equations in terms of cluster pictures:

Theorem 16.3. *Suppose $C : y^2 = f(x)$ is a semistable hyperelliptic curve over K with $f(x) \in \mathcal{O}_K[x]$, and that $|k| > 2g+1$. Then C defines a minimal Weierstrass equation if and only if one of the following conditions hold.*

1. *There are two clusters of size $g+1$ that are swapped by Frobenius, $d_{\mathcal{R}} = 0$ and $v(c) \in \{0,1\}$.*
2. *There is no cluster of size $> g+1$ with depth > 0, but there is some G_K-stable cluster $\mathfrak{s}$ with $|\mathfrak{s}| \geqslant g+1$, $d_{\mathfrak{s}} \geqslant 0$ and $v(c) = -\sum_{r \notin \mathfrak{s}} d_{\{r\} \wedge \mathfrak{s}}$.*

Using examples we now illustrate how one can easily use cluster pictures and the results of this section to check whether a Weierstrass equation is integral and/or minimal.

Example 16.4. Consider $C : y^2 = f(x) = p(x - \frac{1}{p^2})((x - \frac{1}{p^2})^3 - p^9)(x - \frac{1}{p^2} - \frac{1}{p})$, a genus 2 hyperelliptic curve over $\mathbb{Q}_p$, for some prime $p > 3$. Let us use the cluster picture of C to test whether there exists some $z \in K$ such that $f(x-z) \in \mathcal{O}_K[x]$. The cluster picture of C is as follows:



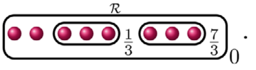
with $d_{\mathcal{R}} = -1$, and $d_{\mathfrak{s}} = 3$.

Note that $\mathcal{R}$ and $\mathfrak{s}$ are both proper and $G_{\mathbb{Q}_p}$-stable, $\mathfrak{s} < \mathcal{R}$, $d_{\mathcal{R}} \leqslant 0$, and $d_{\mathfrak{s}} \geqslant 0$. A simple calculation gives that

$$v(c) + (|\mathcal{R}| - |\mathfrak{s}|)d_{\mathcal{R}} + \sum_{r \notin \mathcal{R}} d_{\{r\} \wedge \mathcal{R}} = 0.$$

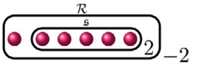
Therefore, by Theorem 16.1, we conclude that there exists some $z \in K$ such that $f(x-z) \in \mathcal{O}_K[x]$. Indeed, we can take $z = -\frac{1}{p^2}$.

Example 16.5. Consider $C : y^2 = (x^2-1)(x^3-p)((x-2)^3-p^7)$, a genus 3 hyperelliptic curve over $\mathbb{Q}_p$, for some prime $p > 3$. The cluster picture of C is as follows:



Note that $d_{\mathcal{R}} = v(c) = 0$ and every cluster $\mathfrak{s} \neq \mathcal{R}$ has size < 4, so by Theorem 16.2 we can conclude that C is a minimal Weierstrass equation.

Example 16.6. Consider $C : y^2 = p^2(x - \frac{1}{p^2})(x^5-1)$, a genus 2 hyperelliptic curve over $\mathbb{Q}_p$ for some prime $p > 5$. The cluster picture of C is as follows:



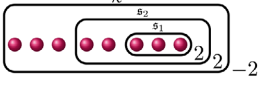
with $d_{\mathcal{R}} = -2$, and $d_{\mathfrak{s}} = 0$.

Note that $d_{\mathcal{R}}, v(c) \neq 0$ and cluster $|\mathfrak{s}| = 5 > 3$, so we are unable to conclude by Theorem 16.2 whether C is a minimal Weierstrass equation. However, one can easily check that the semistability criterion in Section 5 is satisfied (see the examples in that section for further details of how to check this), so C is semistable. Now, there is no cluster of size > 3 with depth > 0, but $\mathfrak{s}$ is $G_{\mathbb{Q}_p}$-stable with $|\mathfrak{s}| = 5 \geqslant 3$, $d_{\mathfrak{s}} = 0$, and $2 = v(c) = -\sum_{\{r\}\notin\mathfrak{s}} d_{\{r\}\wedge\mathfrak{s}}$. So, by Theorem 16.3 we can conclude that C defines a minimal Weierstrass equation.

Example 16.7. Consider the hyperelliptic curve $C : y^2 = (x^3 - p^{15})(x^2-p^6)(x^3-p^3)$ over $\mathbb{Q}_p$ for some prime $p > 7$. We claim that the substitutions $x = p^3x'$ and $y = p^9y'$, result in a minimal Weierstrass equation

$$C' : y'^2 = (x'^3 - p^6)(x'^2-1)(p^6x'^3-1),$$

whose cluster picture is as follows:



with $d_{\mathcal{R}} = -2, d_{\mathfrak{s}_2} = 0$, and $d_{\mathfrak{s}_1} = 2$.

We are able to verify that C' is indeed minimal. Note that its cluster picture has no cluster of size $> g+1$ with depth > 0, but $\mathfrak{s}_2$ is fixed by $G_{\mathbb{Q}_p}$, $|\mathfrak{s}_2| = 5 \geqslant 4$, $d_{\mathfrak{s}_2} = 0$, and $v(c) = -\sum_{r\notin\mathfrak{s}_2} d_{\{r\}\wedge\mathfrak{s}_2} = 6$. So, since C' is semistable, by Theorem 16.3 (2) we have that C' is minimal.

References. 16.1: [9, Theorem 13.3]. 16.2, 16.3: [9, Theorems 17.1, 17.2].

17 | ISOMORPHISMS OF CURVES AND CANONICAL CLUSTER PICTURES

Definition 17.1. Let X be a finite set, Σ a collection of non-empty subsets of X (called *clusters*), and some $d_{\mathfrak{s}} \in \mathbb{Q}$ for every $\mathfrak{s} \in \Sigma$ of size > 1, called the *depth* of $\mathfrak{s}$. Then Σ (or (Σ, X, d)) is a *cluster* *picture* if: $X \in \Sigma$ and $\{x\} \in \Sigma$ for every $x \in X$; two clusters are either disjoint or one is contained in the other; for $\mathfrak{s}, \mathfrak{s}' \in \Sigma$, if $\mathfrak{s}' \subsetneq \mathfrak{s}$ then $d_{\mathfrak{s}'} > d_{\mathfrak{s}}$. For a hyperelliptic curve $C/K : y^2 = f(x)$, denote the *cluster picture* by $\Sigma_C = (\Sigma_C, \mathcal{R}, d)$, the collection of all clusters of $\mathcal{R}$ with depths.

Cluster pictures (Σ^i, X^i, d^i), $i = 1, 2$, are *isomorphic* ($\Sigma^1 \cong \Sigma^2$) if there is a bijection $\phi : X^1 \to X^2$ which induces a bijection from Σ^1 to Σ^2 and $d^1_{\mathfrak{s}} = d^2_{\phi(\mathfrak{s})}$.

Definition 17.2. We say $\Sigma = (\Sigma, X, d)$ and $\Sigma' = (\Sigma', X', d')$ are *equivalent* if Σ' is isomorphic to a cluster picture obtained from Σ in a finite number of the following steps.

(1) *Increase the depth of all clusters by* $m \in \mathbb{Q}$: $d'_{\mathfrak{s}} = d_{\mathfrak{s}} + m$ for all $\mathfrak{s} \in \Sigma$.
(2) *Add a root* r if X is odd: $X' = X \cup \{r\}$, $\Sigma' = (\Sigma \cup \{\{r\}, X'\}) \setminus \{X\}$, $d'_{\mathfrak{s}} = d_{\mathfrak{s}}$ for all proper $\mathfrak{s} \in \Sigma' \setminus \{X'\}$ and $d'_{X'} = d_X$.
(3) *Remove a root* $r \in X$ if X is even, $\{r\} < X$ and $X \setminus \{r\} \notin \Sigma$: $X' = X \setminus \{r\}$, $\Sigma' = (\Sigma \cup \{X'\}) \setminus \{X, \{r\}\}$, $d'_{\mathfrak{s}} = d_{\mathfrak{s}}$ for $\mathfrak{s} \in \Sigma' \setminus \{X'\}$ proper and $d'_{X'} = d_X$.
(4) *Redistribute the depth between child* $\mathfrak{s} < X$ and $\mathfrak{s}^c = X \setminus \mathfrak{s}$ when X is even: pick $m \in \mathbb{Q}$ with $-\delta_{\mathfrak{s}} \leqslant m \leqslant \delta_{\mathfrak{s}^c}$ (if $|\mathfrak{s}| = 1$ there is no lower bound on m, and similarly for $\mathfrak{s}^c$) and set $X' = X$, $\Sigma' = \Sigma \cup \{\mathfrak{s}, \mathfrak{s}^c\}$, $d'_{X'} = d_X$, $d'_{\mathfrak{t}} = d_{\mathfrak{t}} + m$ for proper clusters $\mathfrak{t} \subseteq \mathfrak{s}$, $d'_{\mathfrak{t}} = d_{\mathfrak{t}} - m$ for proper clusters $\mathfrak{t} \subseteq \mathfrak{s}^c$. Here we consider $\delta_{\mathfrak{s}^c} = 0$ if $\mathfrak{s}^c \notin \Sigma$, and remove $\mathfrak{s}^c$ from Σ' if $\delta'_{\mathfrak{s}^c} = 0$.

For a pictorial description of these moves, see Example 17.7.

Theorem 17.3. *If C_1 and C_2 are isomorphic hyperelliptic curves over K, then their cluster pictures are equivalent. Furthermore, if a cluster picture Σ' is equivalent to Σ_{C_1}, then there is a $\bar{K}$-isomorphic hyperelliptic curve $C'/\bar{K}$ with $\Sigma_{C'} \cong \Sigma'$.*

Theorem 17.4. *Let C_1 and C_2 be semistable hyperelliptic curves over K. Then Σ_{C_1} and Σ_{C_2} are equivalent if and only if the BY trees T_{C_1} and T_{C_2} are isomorphic.*

It turns out that, provided $|k| > 2g + 1$, every equivalence class of cluster pictures of semistable hyperelliptic curves has an 'almost canonical' representative.

Theorem 17.5. *Let C'/K be a semistable hyperelliptic curve and suppose that $|k| > 2g + 1$. Then there is a K-isomorphic curve $C : y^2 = f(x)$ with $f(x) \in \mathcal{O}_K[x]$, $\deg(f) = 2g + 2$ such that:*

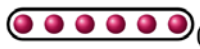
(1) $d_{\mathcal{R}} = 0$*;*
(2) *the cluster picture of C has no cluster of size $> g + 1$ other than $\mathcal{R}$; and*
(3) *either there is at most one cluster in Σ_C of size $g + 1$ and $v(c) = 0$, or* Frob *swaps two clusters of size $g + 1$ and $v(c) \in \{0, 1\}$.*

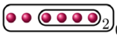
Furthermore, if C' has even genus, then we may replace (3) by the following.

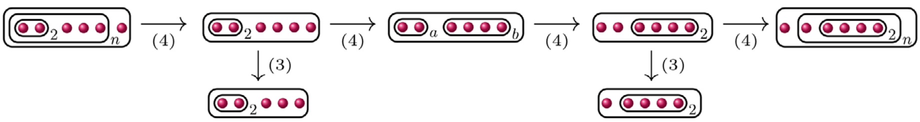
(3') *either $v(c) = 0$ and there is no cluster of size $g + 1$, or $v(c) \in \{0, 1\}$ and there are two clusters of size $g + 1$ with equal depths.*

In the even genus case, any other K-isomorphic curve satisfying (1), (2), and (3') has the same cluster picture and valuation of leading term as C.

For a semistable hyperelliptic curve C/K, to practically use BY trees to find the canonical representative of the equivalence class of Σ_C, attach an open yellow edge to the *centre* ([10, Definition 5.13]) of T_C. For a more detailed explanation of this, see Remarks A.8 and A.9.

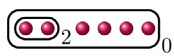
Example 17.6. Consider the hyperelliptic curve $C : y^2 = x^6 - 1$ over $\mathbb{Q}_p$, for some prime $p \neq 3$, where $\Sigma_C =$ $_0$. By Definition 17.2 (1), we may increase the depth of $\mathcal{R}$ by $m = \frac{1}{3}$ to obtain an equivalent cluster picture. Theorem 17.3 tells us there is some $\bar{\mathbb{Q}}_p$-isomorphic curve $C'/\bar{\mathbb{Q}}_p$ with this cluster picture. In particular, we find that under the transformations $x = x'/p^{1/3}$ and $y = y'/p$, C is $\mathbb{Q}_p(\sqrt[3]{p})$-isomorphic to $C'/\mathbb{Q}_p(\sqrt[3]{p}) : y'^2 = x'^6 - p^2$.

Example 17.7. Consider the hyperelliptic curve $C/\mathbb{Q}_7 : y^2 = (x^2-1)(x^4-7^8)$. It has cluster picture $\Sigma_C =$ $_0$ with $\mathcal{R} = \{1, -1, 7^2, -7^2, 7^2 i, -7^2 i\}$. Definition 17.2 gives us that the equivalence class of Σ_C is as follows:



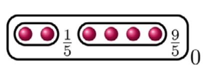
Here the top clusters' depths are not written as these can take any value, due to Definition 17.2 (1), and $n, a, b \in \mathbb{Q}_{>0}$ with $a + b = 2$. Vertical lines indicate that a root has been added or removed as in Definition 17.2 (2) and (3). Horizontal lines indicate that the depth of a child $\mathfrak{s} < \mathcal{R}$ has been redistributed to $\mathcal{R} \setminus \mathfrak{s}$ as described in Definition 17.2 (4).

Let $C_1/\mathbb{Q}_7 : y^2 = (x^2 - 7^4)(x^4 - 1)$, this is isomorphic to C over $\mathbb{Q}_7$ and has

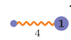
$\Sigma_{C_1} =$ $_0$.

So, Σ_{C_1} is in the equivalence class of Σ_C, verifying the first part of Theorem 17.3.

Consider the transformation $x \to \frac{\sqrt[5]{7}}{x+\sqrt[5]{7}}$. It gives a model C_2 for $C/\mathbb{Q}_7(\sqrt[5]{7})$ with roots $\frac{\sqrt[5]{7}}{1+\sqrt[5]{7}}, \frac{\sqrt[5]{7}}{-1+\sqrt[5]{7}}, \frac{1}{1+\sqrt[5]{7}^9}, \frac{1}{1-\sqrt[5]{7}^9}, \frac{1}{1+i\sqrt[5]{7}^9}, \frac{1}{1-i\sqrt[5]{7}^9}$, and cluster picture

$_0$.

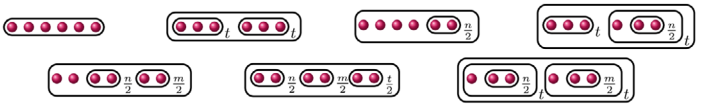
This illustrates how to obtain the middle picture with $a = \frac{1}{5}$ and $b = \frac{9}{5}$ over $\bar{\mathbb{Q}}_7$.

All of C, C_1, and C_2 have the following BY tree: . Indeed, so does any other hyperelliptic curve with a cluster picture in the equivalence class of Σ_C. Conversely, any hyperelliptic curve C' with BY tree $T_{C'} = T_C$ would need to have its cluster picture in the equivalence class of Σ_C.

Remark 17.8. It is useful to note that the steps described in Definition 17.2 can be made by applying the following Möbius transformations to the roots in $\mathcal{R}$:

(1) $\phi(z) = \pi^m z$ (for $m \in \mathbb{Q}$),
(2) $\phi(z) = \frac{1}{z}$ (after first shifting by $z_{\mathcal{R}} \in K$, that is, applying $\phi'(z) = z - z_{\mathcal{R}}$),
(3) $\phi(z) = \frac{1}{z}$ (first shifting by r and using (1) to assume that $z_{\mathcal{R}} = r = d_{\mathcal{R}} = 0$),
(4) $\phi(z) = \frac{\pi^a}{z}$ (first scaling so $d_{\mathcal{R}} = 0$, and shifting so $v(r) = a$ for $r \in \mathfrak{s}$).

Example 17.9. By Theorem 17.5, any semistable genus 2 hyperelliptic curve, where $|k| > 2g+1$, has a model with one of the following cluster pictures with $m, n, t \in \mathbb{Z}$:



References. 17.2, 17.3, 17.5: [9, Sections 14 and 15]. 17.4: [10, Sections 4.2, 5.2]. 17.8: [9, Proposition 14.6].

APPENDIX A: MINIMAL DISCRIMINANT AND BY TREES (SEMISTABLE CASE)

Throughout this section, it is assumed that C is *semistable*. We give a proof for how to read off $v(\Delta_C^{\min})$ from the BY tree T_C associated to C.

Definition A.1. For a connected subgraph T of a BY tree, we define a genus function by $g(T) = \#(\text{connected components of the blue part}) - 1 + \sum_{v \in V(T)} g(v)$.

Note that $g(T_C) = g$ as per Lemma 4.8.

Definition A.2. If there is an edge $e \in E(T_C)$ such that both trees in $T_C \setminus \{e\}$ have equal genus (that is, genus $\lfloor \frac{g}{2} \rfloor$), then we insert a genus-0 vertex z_T on the midpoint of e, colour it the same as e, and call it the *centre* of T_C. Otherwise, choose $z_T \in V(T_C)$ such that all trees in $T_C \setminus \{z_T\}$[†] have genus smaller than $g/2$. In both cases, the *centred BY tree* T_C^* is the tree with vertex set $V(T_C^*) = V(T_C) \cup \{z_T\}$; we denote by $\preceq$ the partial order on $V(T_C^*)$ with maximal element z_T. For a vertex $v \in V(T_C^*)$, we say that the vertex connected to v lying on the path to the centre of T_C^* is its *parent*. All other vertices connected to v are called *children* of v. The centre itself does not have a parent.

Definition A.3. Define a weight function on the vertex set $V(T_C)$ by

$$s(v) = \begin{cases} 2g(v) + 2 - \#\text{blue edges at } v & \text{if } v \text{ is blue,} \\ 0 & \text{if } v \text{ is yellow.} \end{cases}$$

For a connected subgraph T of T_C, we set $s(T) = \sum_{v \in T} s(v)$.

Remark A.4. Observing that $s(T_C) = 2g + 2$, it follows from [10, Lemma 5.12] that exactly one of the following is true.

- There is a unique vertex $v \in V(T_C)$ with the property that $s(T) < g + 1$ for all trees in $T_C \setminus \{v\}$.
- There is a unique edge $e \in E(T_C)$ with the property that $s(T) = g + 1$ for both trees in $T_C \setminus \{e\}$.

Further, $g(T) = \lfloor \frac{s(T)-1}{2} \rfloor$ for any connected subgraph T of a BY tree (see [10, Remark 5.14]). This shows that the centre of a BY tree is indeed well defined.

Definition A.5. Define a weight function on $V(T_C^*)$ by $S(v) = \sum_{v' \preceq v} s(v')$.

[†] $T_C \setminus \{z_T\}$ is obtained from T_C by removing z_T together with the incident edges.

For each $v \neq z_T$, write e_v for the edge connecting v with its parent and let

$$\delta_v = \begin{cases} \text{length}(e_v) & \text{if } e_v \text{ is blue,} \\ 1/2 \cdot \text{length}(e_v) & \text{if } e_v \text{ is yellow.} \end{cases}$$

Theorem A.6. *Let T_C^* be the centred BY tree associated to C. Suppose $|k| > 2g+1$. Then the valuation of the minimal discriminant of C is given by*

$$v(\Delta_C^{\min}) = E \cdot (4g+2) + \sum_{v \neq z_T} \delta_v S(v)(S(v)-1),$$

where $E = 0$ unless z_T has exactly two children v_1, v_2 with $S(v_1) = S(v_2) = g+1$ that are permuted by Frobenius and $\delta_{v_i}(g+1)$ is odd for $i \in \{1,2\}$. In this case $E = 1$.

Proof. Let $\Sigma = \Sigma_C$ be the cluster picture associated to C, see Definition 17.1 for the definition of abstract cluster pictures. We associate a cluster picture $\Sigma_1 = (\Sigma_1, X_1, d_1)$ to the centred tree T_C^* in the following way.

For every vertex $v \in T_C^*$, define

$$\mathfrak{s}_v = \bigcup_{v' < v \text{ maximal}} \mathfrak{s}_{v'} \cup \bigcup_{i=1}^{s(v)} \{r_{v,i}\},$$

where $\{r_{v,i}\}$ are singletons. For $v \neq z_T$, the relative depth of the cluster $\mathfrak{s}_v$ is given by $\delta_{\mathfrak{s}_v} = \delta_v$. We have $\mathfrak{s}_{z_T} = X_1$ and assign to it depth $d_{X_1} = 0$.

The construction of the cluster picture coincides with Construction 4.15 in [10], although phrased in a slightly different language (cf. Remark A.9). Therefore, the BY tree associated to this cluster picture is T_C. Moreover, it is clear from the construction that for every vertex $v \in V(T_C^*)$, we have $S(v) = |\mathfrak{s}_v|$ and that every cluster $\mathfrak{s} \neq \mathcal{R}$ has size $\leqslant g+1$.

From Theorems 17.3 and 17.4, it follows that there is a hyperelliptic curve $C_1 : y^2 = f_1(x)$ which is $\bar{K}$-isomorphic to C and has cluster picture Σ_1. Applying the formula of Theorem 15.1, we find that

$$v(\Delta_{C_1}) = v(c_1)(4g+2) + \sum_{v \neq z_T} \delta_v S(v)(S(v)-1), \tag{A.1}$$

where c_1 denotes the leading coefficient of f_1. We will now modify the cluster picture Σ_1 in order to find a curve C_2 which is isomorphic to C over K.

Let us first consider the case where $z_T \in V(T_C)$. In that case, we moreover have that $|\mathfrak{s}_v| < g+1$ for all clusters $\mathfrak{s}_v \neq \mathcal{R}$. It follows from Theorem 17.5 and the uniqueness of the centre z_T that there is a K-isomorphic curve $C_2 : y^2 = f_2(x)$ with cluster picture $\Sigma_{C_2} = \Sigma_1$ and $v(c_2) = 0$, where c_2 is the leading coefficient of f_2. This completes the first case.

Now consider the case $z_T \notin V(T_C)$. Then $\mathcal{R} = \mathfrak{s}_1 \sqcup \mathfrak{s}_2$, where $|\mathfrak{s}_1| = |\mathfrak{s}_2| = g+1$. In this case, it might be necessary to redistribute depth between the clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$, see Definition 17.2. However, this does not change the valuation of the discriminant since the two clusters have equal size. Hence, we may still use equation (A.1). If the two clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are not permuted by Frobenius, let Σ_2 be the cluster picture obtained by redistributing all depth from $\mathfrak{s}_1$ to $\mathfrak{s}_2$ (or vice versa). It follows from 17.5 that there is a K-isomorphic curve C_2 with this cluster picture and $v(c_2) = 0$.

In the other case, where the two clusters $\mathfrak{s}_1, \mathfrak{s}_2$ are permuted by Frobenius, we know that there exists a curve C_2 which is isomorphic to C with $v(c_2) \in \{0, 1\}$ and $\Sigma_{C_2} = \Sigma_2$, where Σ_2 is obtained from Σ_1 by shifting depth $m \in \mathbb{Q}$ from $\mathfrak{s}_1$ to $\mathfrak{s}_2$. It remains to compute $v(c_2)$. For that purpose denote by $\delta_1 = \delta_{\mathfrak{s}_1} - m$ and $\delta_2 = \delta_{\mathfrak{s}_2} + m$ the new relative depths of the clusters $\mathfrak{s}_1$ and $\mathfrak{s}_2$. It follows from the semistability criterion (Theorem 5.1) that $v(c_2) \equiv \delta_1(g+1) \equiv \delta_2(g+1) \pmod 2$. If g is odd, this implies $v(c_2) = 0$. On the other hand, if g is even, we may assume that $\delta_1 = \delta_2$ (see Theorem 17.5). Hence, $v(c_2) = 1$ if and only if $\delta_{\mathfrak{s}_i}(g+1)$ is odd.

In all cases, we have seen that there is a K-isomorphic curve for which $v(c_2) = E$ and the valuation of the discriminant is given by the formula in the theorem. By Theorem 15.2, this is indeed the valuation of the minimal discriminant. □

Remark A.8. The cluster picture Σ_1 constructed in the proof presents a canonical representative for the equivalence class of the cluster picture associated to C (see Definition 17.2).

Remark A.9. Instead of working with the centred BY tree T_C^*, one could also consider the open BY tree [10, Definition 3.21] obtained by gluing an open yellow edge to the centre of T_C. The order on the vertices of this tree and the construction of the cluster picture Σ_1 described in the proof of the theorem then coincide exactly with the definitions in Construction 4.15 in [10].

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